Two-dimensional plasma flow past a laser beam

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Analytical results are presented for laser beam deflection rate due to plasma flow when the ponderomotive force (PMF) is static and given. Explicit expressions are obtained in various parameter regimes including that of weak PMF for the case of a coherent (diffraction limited) beam and a beam whose fluctuations are spatially homogeneous, as in the case of a model random phase plate beam. When the Landau damping coefficient, γ_0, is negligible and the beam is either coherent and cylindrically symmetric, or random with isotropic fluctuations, the deflection rate is obtained as a closed form function of plasma flow Mach number, M. For finite damping, results are expressed in terms of a universal, one dimensional integral parameterized by M and γ_0. For arbitrary PMF and M small, the problem is identified with one in the theory of random dielectric media. © *1997 American Institute of Physics.* [S1070-664X(97)00307-8]

I. INTRODUCTION

Laser beam pointing in hohlraum plasmas has been observed not to agree with model predictions which only allow for refraction.[1,2] It has been shown[3] that random phase plate (RPP) conditioned laser beams[4] and diffraction limited beams are deflected in the direction of transverse plasma flow, consistent with hohlraum[5] and exploding foil[6,7] experiments. While theoretical studies[8,3] have shown that beam deflection follows from general principles, and numerical simulations to evaluate this effect in special cases have been presented, there has been no quantitative theory. Even for the reduced problem, where the ponderomotive force of the laser beam is that of a given isolated hot spot, only a few isolated analytical results for special cases are available in the literature.[9,3,8] In this paper analytical results are presented, over finite parts of the parameter space defined by the flow Mach number and ponderomotive force, for the laser beam deflection rate for a model in which the ponderomotive force is given. In particular, closed form analytic expressions which relate beam deflection to the ponderomotive force correlation function are obtained for linear hydrodynamics.

In the next section we introduce a simple model problem formulated in terms of classical compressible, inviscid fluid dynamics, to describe the flow of a neutral plasma past a laser beam of known characteristics. The model involves two parameters, the Mach number of the inflow, and a parameter proportional to the strength of the ponderomotive force. We then present, in Sec. III, asymptotic solutions to this model, in the two parameter regimes corresponding to low Mach number, and weak ponderomotive force respectively. Explicit formulas are presented for the beam deflection in terms of the beam intensity profile as well as three ''null deflection theorems'' specifying the regions of parameter space where no deflection is possible. In Sec. IV we examine how the introduction of a dissipation mechanism, namely, Landau damping changes the beam deflection results of the previous section. In hohlraum and exploding foil experiments, the laser beam intensity profiles are not smooth, but have random fluctuations. This is considered in Sec. V. In the limit of a weak ponderomotive force an expression for the beam deflection is derived in terms of the two point correlation function of the ponderomotive potential. In the limit of low Mach number, it is shown that the beam deflection problem is equivalent to a problem involving a random dielectric medium. Some partial solutions are presented for simplified models. A summary of results, possible directions for future research, and an assessment of the possible uses of these results in hohlraum related technology is provided in Sec. VI.

II. FORMULATION OF A SIMPLE MODEL PROBLEM

Let us consider a neutral plasma with a flow velocity U_* in the x-direction in regions far from the origin. Near the origin, there is an electromagnetic wave propagating in the z-direction with electric field $\mathbf{E}=\Re[\mathbf{E}_* e^{i(\omega t-kz)}]$, where the complex amplitude $\mathbf{E}_*$ varies in x and y over some characteristic distance a, and varies only weakly in the z-direction. When the electrons in the plasma are subjected to such an oscillating field of variable amplitude, a mean drift is created that tends to transport the electrons from regions of high to low field intensities. In a coarse grained description the plasma behaves as if it were subjected to a force density whose potential per unit mass of plasma is given by

$$V=\frac{Z}{m_i}U=\frac{Ze^2}{4m_i m_e\omega^2}|E_*|^2\equiv V_*\Omega(\mathbf{x}/a), \tag{1}$$

where m_e and m_i are the electron and ion masses respectively, e is the electronic charge, Z is the charge state and U is the ponderomotive potential. The ponderomotive force may lead to a depletion of plasma density near the origin which causes intensification of the electromagnetic wave due to focusing, as in an optic fiber, this in turn leads to a further depletion of density. This self-focusing instability may cause

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a smooth beam to filament or nonlinearly self-focus preexisting local intensity maxima (or hot-spots in the case of a nonsmooth beam) increasing the probability of high intensity fluctuations.[10–12] In this paper we would like to study the effect of a steady transverse flow on the direction of propagation of the laser beam as a first step towards studying the more general problem of time dependent self-focusing in the presence of a transverse flow. Therefore we will simplify the problem by ignoring the feedback effect of the perturbation of plasma density on the electromagnetic wave. Thus, the ponderomotive potential would be assumed to be known as a function of x and y. Since time scales of interest are much longer than the oscillation period of Langmuir waves, and since the plasma parameter $g=1/n\lambda_D^3$ (n is the electron number density and λ_D is the Debye length) is small, the plasma can be treated as an ordinary compressible neutral gas. In the physical applications of interest, the thermal conductivity of the plasma is very high so that the gas can be assumed isothermal. The sound speed c_s is therefore constant. Similarly, the viscosity of the plasma can be neglected. In the absence of viscosity, the dominant dissipation mechanism is the Landau damping of ion-acoustic waves. This latter dissipation mechanism is small for a large class of problems and we will temporarily neglect it. Its effect is investigated in Sec. IV. Since the fluid is assumed inviscid and isothermal, there is no generation of vorticity in the flow. We may therefore introduce the velocity potential ϕ defined by $\mathbf{u}=U_* a\nabla\phi$ where $\mathbf{u}(\mathbf{x})$ is the flow velocity. We now introduce the dimensionless parameters, $M=U_*/c_s$, the Mach number, and $S=V_*/c_s^2$, characterizing the strength of the ponderomotive potential.

The basic equations are, Bernoulli's equation:

$$\ln\rho=\frac{M^2}{2}(1-|\nabla\phi|^2)-S\Omega, \tag{2}$$

and the continuity equation:

$$\nabla^2\phi+\frac{\nabla\phi\cdot\nabla\rho}{\rho}=0, \tag{3}$$

where ρ is the density normalized by the density of the unperturbed flow far from the origin, ρ_*, and all variables with the physical dimension of length have been scaled by a. From this point on we will only use dimensionless variables. Any exceptions to this rule would be indicated by naming the corresponding variable with a suffix *. On eliminating ρ between Eqs. (2) and (3) we obtain

$$\nabla^2\phi-S\nabla\phi\cdot\nabla\Omega=\frac{M^2}{2}\nabla\phi\cdot\nabla(|\nabla\phi|^2), \tag{4}$$

which is the basic equation we need to solve, together with the boundary condition, that, the flow reduce to the unperturbed incident flow far upstream;

$$\phi\overset{x\to-\infty}{\sim}x. \tag{5}$$

The beam deflection is in the direction of the flow, which is the x-direction. In response to the ponderomotive force, the direction of propagation of the laser light bends away from the z-axis in the x-z plane. If $\alpha(z_*)$ denotes the angle between the beam direction and the z-axis, the rate of change of α with z_* is given by[3]

$$a\frac{d\alpha}{dz_*}=\frac{\rho_*}{2\rho_c}\frac{\int\rho\partial_x\Omega dxdy}{\int\Omega dxdy}=-\frac{\rho_*}{2\rho_c}\frac{\int\Omega\partial_x\rho dxdy}{\int\Omega dxdy}, \tag{6}$$

where ρ_c is the critical plasma density and the suffix * in z_* indicates that this variable is in physical units.

III. ASYMPTOTIC SOLUTIONS

The basic equation of our model, Eq. (4), has different regimes of qualitative behavior in various regions of the parameter space (M,S). Solutions for arbitrary M and S can be obtained only through numerical computation. In order to make analytical progress we will restrict ourselves to seeking asymptotic solutions in the limits $M\to 0$, at a fixed S (''Flow at low Mach number'') and $S\to 0$, at a fixed M (''Flow past a weak ponderomotive potential''). By studying the asymptotic behavior along these two axes in parameter space, we hope to be able to construct a qualitative understanding of the structure of the whole parameter space. We divide this section into two subsections dealing with the $M\to 0$ and $S\to 0$ cases respectively. The latter is further subdivided into three parts dealing with subsonic flow ($M<1$), supersonic flow ($M>1$) and transonic flow ($M\approx 1$).

In Sec. III A the beam deflection is shown to be zero up to terms of order M^2, for small M, independent of the shape of Ω. However, explicit analytical solutions are presented for special forms of Ω in the limit of small M, as these solutions will be used later in Sec. III B3 and Sec. IV A. In Sec. III B 1, the beam deflection is again shown to be generally zero to lowest order in S for subsonic flow ($M<1$), independent of the shape of Ω. However, explicit analytical solutions are also presented for special forms of Ω, as they are used later in Sec. III B 3 to compute the boundaries of the transonic regime in parameter space. In Sec. III B 2, a general expression for beam deflection is derived for an arbitrary potential in the limit of small S but $M>1$. Explicit solutions for the flow are presented for special shapes of Ω for use in Sec. III B 3. A general theorem that establishes the conditions under which nonzero beam deflection is possible is stated and proved for any potential Ω that has the symmetry $\Omega(-x,y)=\Omega(x,y)$. Sec. III B 3 describes the special regime of ''transonic'' flow (S small, but $M\approx 1$). In this regime the basic flow equations are intrinsically nonlinear and consequently very difficult to solve analytically. Analytical results are presented only for the boundary of the transonic regime in parameter space for special forms of the potential Ω.

A. Flow at low Mach number

In the limit $M\to 0$, the solution, ϕ, of Eqs. (4) and (5) may be sought as an asymptotic series in the small parameter M^2;

$$\phi=\phi_0+M^2\phi_1+M^4\phi_2+\cdots. \tag{7}$$

On substituting Eq. (7) in Eqs. (4) and (5) we obtain at the lowest order

$$\nabla^2\phi_0 - S\nabla\phi_0\cdot\nabla\Omega = 0 \tag{8}$$

with the boundary condition

$$\phi_0 \overset{x\to-\infty}{\sim} x. \tag{9}$$

Let us define a new variable

$$\kappa = \exp(-S\Omega). \tag{10}$$

Then Eq. (8) may be written as

$$\nabla\cdot(\kappa\nabla\phi_0) = 0. \tag{11}$$

Equations (11) and (9) are formally equivalent to the problem of determining the potential due to a dielectric medium localized near the origin, with dielectric coefficient $\kappa(x,y)$, and a uniform electric field of unit magnitude directed opposite to the x-axis far from the origin. We will make repeated use of this formal equivalence which will be referred to as ''the dielectric analogy.'' The only difference with a physical dielectric medium is that, the ''dielectric coefficient'' $\kappa=\exp(-S\Omega)<1$ which corresponds to an unphysical ''negative electric susceptibility.'' In physical dielectrics, molecules tend to develop a dipole moment in the direction of the applied field so that the susceptibility is positive.

An explicit analytical solution to Eqs. (8) and (9) can be provided only for special forms of Ω. In particular, we will use the following form which we will refer to as the ''step'' potential:

$$\Omega_{\text{step}}(r,\theta) = \begin{cases} 1 & \text{if } r\leqslant 1, \\ 0 & \text{if } r>1, \end{cases} \tag{12}$$

where (r,θ) are polar coordinates in the $x-y$ plane. In terms of the dielectric analogy, this problem is formally equivalent to the problem of an infinite, uniform dielectric cylinder placed in a uniform electric field, the solution of which is well known.[13] This enables us to write down the solution to Eqs. (8) and (9) by simply replacing the dielectric constant κ in the above solution by the expression (10):

$$\phi_0 = \phi_0^{(\text{step})} = \begin{cases} \dfrac{2}{1+\exp(-S)}\, r\cos\theta & \text{if } r\leqslant 1, \\ \left[r + \dfrac{1}{r}\tanh\left(\dfrac{S}{2}\right)\right]\cos\theta & \text{if } r>1. \end{cases} \tag{13}$$

When S is large, $\phi_0 \sim (r+1/r)\cos\theta$ for $r>1$, which is identical to the classical potential flow around a rigid cylindrical obstacle. However, when $r\leqslant 1$, the solution in the limit of large S is $\phi_0 \sim 2r\cos\theta$, which corresponds to uniform flow inside the cylinder in the same direction as that of the far field but with twice the magnitude. This internal flow originates from the requirements of continuity of the tangential component of the velocity and the normal component of $\kappa\nabla\phi_0$. The discontinuity in the normal component of the velocity is to be expected from considerations of momentum conservation since the force density is infinite at the boundary of the step potential. When S is small, $\phi_0 \sim (r+S/2r)\cos\theta$ for $r>1$ and $\phi_0 \sim (1+S/2)r\cos\theta$ for $r\leqslant 1$.

This step model for the potential is somewhat unphysical, because it implies an infinite force density. It will be shown later that this leads to infinite beam deflections and velocity fields that are singular along certain curves. We therefore introduce the following ''parabolic'' potential model that is more physically reasonable:

$$\Omega_{\text{par}}(r,\theta) = \begin{cases} 1-r^2 & \text{if } r\leqslant 1, \\ 0 & \text{if } r>1. \end{cases} \tag{14}$$

The force density in this case is finite everywhere though it changes discontinuously at $r=1$. This however does not result in anything catastrophic such as infinite beam deflections as we shall see. The model is however harder to solve analytically, and we are able to solve it only in the two limits $S\to 0$ and $S\to\infty$. The former is already covered by the solutions presented in Sec. III B, here we consider the latter. The solution for $S\to\infty$ can be given for a general radially symmetric potential up to certain quadratures. The form (14) is assumed only for evaluating the integrals explicitly. Equation (8) is separable in the variables r and θ. Further, the boundary condition (9) implies that the solution must be of the form $A(r)\cos\theta$ where $A(r)$ satisfies

$$\frac{d^2A}{dr^2} + \left[\frac{1}{r} - S\Omega'(r)\right]\frac{dA}{dr} - \frac{A}{r^2} = 0. \tag{15}$$

When S is large, Eq. (15) has the nature of an oscillator equation with large damping, in which case, the ''inertia term'' A'' may be dropped:

$$[1 - Sr\Omega'(r)]\frac{dA}{dr} - \frac{A}{r} = 0. \tag{16}$$

In the coefficient of A', the first term must be retained even though it is of order S^{-1} compared to the second. This is because it becomes the dominant term as $r\to 0$. Near the origin, the exact Eq. (15) takes the form

$$\frac{d^2A}{dr^2} + \frac{1}{r}\frac{dA}{dr} - \frac{A}{r^2} = 0, \tag{17}$$

which admits two linearly independent solutions r and $1/r$. The requirement of finiteness at the origin selects the first of these solutions which satisfies the equation $A' - A/r = 0$, and this is identical to the approximate Eq. (16) near the origin. Equation (16) is easily integrated,

$$A(r) = A(1)\exp\left[\int_1^r \frac{dR}{\{R - SR^2\Omega'(R)\}}\right], \tag{18}$$

where $A(1)$ must be chosen so as to satisfy the boundary condition $A(r)\to r$ as $r\to\infty$. The integral can be evaluated in closed form for the parabolic potential, Eq. (14), and we obtain

$$A(r) = A(1)(1+2S)^{1/2}\frac{r}{\sqrt{1+2Sr^2}}, \tag{19}$$

for $r\leqslant 1$. For $r>1$, $\Omega=0$, and Eq. (15) then has the solution (with the boundary condition $A\sim r$ as $r\to\infty$)

$$A(r) = r + \frac{C}{r}. \tag{20}$$

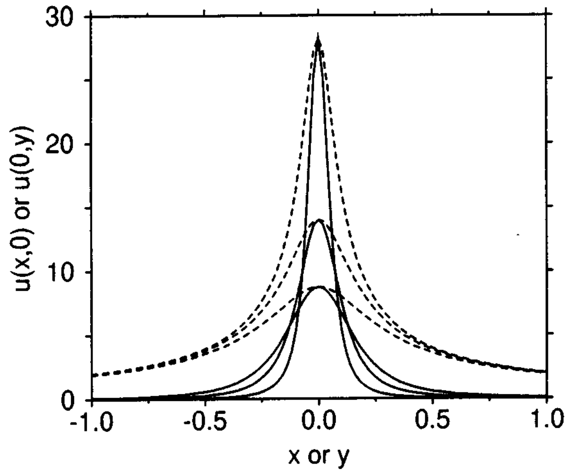

FIG. 1. Velocity profiles, $u(x,0)$ (solid line) and $u(0,y)$ (dashed line), for the ''parabolic'' potential, $\Omega_{\rm par}$, for $S=100$ (highest peak value), 25 (intermediate peak value) and 10 (lowest peak value) with $\gamma_0=0$.

We now require that A and A' match up at $r=1$. This gives two equations which are solved to obtain C and $A(1)$. The result is

$$A(r)=A_{\rm par}(r)=\begin{cases}\dfrac{(1+2S)^{3/2}}{1+S}\dfrac{r}{\sqrt{1+2Sr^2}} & \text{if } r\leqslant 1,\\[2ex] r+\dfrac{S}{1+S}\dfrac{1}{r} & \text{if } r>1.\end{cases} \tag{21}$$

When $S\to\infty$, the flow outside reduces to the potential flow around a rigid cylinder $A=r+1/r$ as in the case of the step potential. However, certain qualitative aspects of this solution are different from that of the step potential and are probably more generic. First, we obtain the x-component of the velocity, along the x-axis and y-axis in the interior region $r<1$:

$$u(x,0)=\frac{(1+2S)^{3/2}}{1+S}\frac{1}{(1+2Sx^2)^{3/2}}, \tag{22}$$

$$u(0,y)=\frac{(1+2S)^{3/2}}{1+S}\frac{1}{(1+2Sy^2)^{1/2}}. \tag{23}$$

Figure 1 shows $u(x,0)$ and $u(0,y)$ for several values of S. It is clear, that the maximum velocity is at the center, and is of order $\sqrt{S}$. This large velocity is reached in a small neighborhood at the center, of radius of the order of $1/\sqrt{S}$.

Let us now compute the beam deflection from Eq. (6) for small Mach numbers. We expand the density in an asymptotic series $\rho=\rho_0+M^2\rho_1+\cdots$ and substitute in Eq. (2) together with the corresponding expansion (7) for ϕ. On equating like powers of M, we get,

$$\rho_0=\exp(-S\Omega) \tag{24}$$

and

$$\rho_1=\frac{1}{2}\exp(-S\Omega)[1-|\nabla\phi_0|^2]. \tag{25}$$

On substituting these expressions into Eq. (6) and performing several integration by parts we obtain the following expression for the beam deflection

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=-\frac{M^2}{4S}\frac{\int\exp(-S\Omega)\partial_x(|\nabla\phi_0|^2)dxdy}{\int\Omega dxdy}. \tag{26}$$

We now show that this quantity is actually zero for an arbitrary potential Ω, that is, in the limit of low Mach numbers and in the absence of Landau damping, no beam deflection is possible.

Theorem 1: In the limit of small M, the beam deflection is zero up to terms of order M^2 for fixed S.

Proof: The proof becomes a standard result of electrostatics if we invoke the ''dielectric analogy.'' Since in this limit the perturbation in the potential decays far from the source, we have,

$$\frac{1}{4\pi}\int\exp(-S\Omega)\nabla(|\nabla\phi_0|^2)dxdy$$

$$\equiv\frac{1}{4\pi}\int\kappa\nabla E^2dxdy=\int\frac{\kappa-1}{4\pi}\nabla E^2dxdy\equiv\mathbf{f}, \tag{27}$$

where $\mathbf{E}=-\nabla\phi_0$ is the ''electric field.'' The quantity, $\mathbf{f}$ is simply the ''total force'' acting on the ''dielectric'' and since the far field is uniform, $\mathbf{f}$ must be zero. The validity of this claim can be demonstrated as follows. Indeed, if we use the facts that $\mathbf{E}$ is constant far from the origin, $\mathbf{E}$ is irrotational $(\partial_iE_j=\partial_jE_i)$, and $\partial_i(\kappa E_i)=0$, (summation implied over repeated tensor indices) we have

$$\begin{aligned}4\pi f_i&=\int\kappa\partial_i(E_jE_j)dxdy\\&=2\int\kappa E_j\partial_iE_jdxdy\\&=2\int\kappa E_j\partial_jE_idxdy\\&=-2\int E_i\partial_j(\kappa E_j)dxdy=0.\end{aligned} \tag{28}$$

(Q.E.D.)

The above result is essentially ''D'Alembert's paradox'' of classical hydrodynamics[14] in a slightly more general context. ''D'Alembert's paradox'' is the result that an irrotational, incompressible, and inviscid fluid flowing past a rigid obstacle exerts no net force on it.

B. Flow past a weak ponderomotive potential

We now consider the limit $S\to 0$ (but M fixed). Physically this means that the ponderomotive force is weak so that the flow deviates only slightly from the uniform flow far from the origin. Thus, we may write the flow potential ϕ as

$$\phi=x+S[\psi_0+S\psi_1+S^2\psi_2+\cdots]. \tag{29}$$

Substitution in Eq. (4) gives at the lowest order

$$(1-M^2)\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}=\frac{\partial\Omega}{\partial x} \tag{30}$$

with the boundary condition $\psi_0 \to 0$ as $x \to -\infty$. This is the basic equation of linearized hydrodynamics. There are three distinct regimes based on the qualitative character of the solution, the subsonic regime ($M<1$), the supersonic regime ($M>1$) and the transonic regime ($M=1$) that we now consider. In the following three subsections we will use the notation $\psi \equiv S\psi_0$.

1. Subsonic flow

We consider Eq. (30) with $M<1$. It is convenient to introduce $\beta=\sqrt{1-M^2}$ and a new stretched variable $\xi \equiv x/\beta$ in place of x. Equation (30) then takes the form

$$\psi_{\xi\xi}+\psi_{yy}=\frac{S}{\beta}\frac{\partial \Omega}{\partial \xi}, \tag{31}$$

which is Poisson's equation. With the boundary condition $\psi \to 0$ as $\xi \to -\infty$, it has the solution

$$\psi(\xi,y)=\frac{S}{2\pi\beta}\int \frac{\partial \Omega(\xi',y')}{\partial \xi'} \ln[(\xi-\xi')^2 + (y-y')^2]^{1/2} d\xi' dy'. \tag{32}$$

A solution in closed form is possible when Ω is the step potential (12). This special solution is now presented. The solution, ψ, of Eq. (31) is identical to the electric potential of a charge density $-S\Omega_\xi/(4\pi\beta)$. When $\Omega=\Omega_{\text{step}}$, the step potential, this ''charge density'' is nonzero only on the surface of the ellipse $\beta^2\xi^2+y^2=1$. Thus, ψ is identical to the potential of a charge, distributed on the surface of a cylinder of elliptic cross section, with surface charge density that is easily shown to be $(S\beta/4\pi)\xi/\sqrt{y^2+\beta^4\xi^2}$. This is a standard electrostatics problem that can be readily solved by separation of variables in elliptic coordinates (u,v) defined by

$$\xi=a \cosh u \cos v, \tag{33}$$

$$y=a \sinh u \sin v, \tag{34}$$

where $\infty>u\geqslant 0$ and $2\pi>v\geqslant 0$. The constant a must be chosen such that the elliptical level line $u=u_0$, for some u_0 coincides with the surface of the charged ellipse. This requires

$$a=M/\beta, \tag{35}$$

$$\tanh u_0=\beta. \tag{36}$$

The solution ψ then appears in the form

$$\psi=S\sum_{n=0}^{\infty} \alpha_n A_n(u)\cos[(2n+1)v], \tag{37}$$

where

$$A_n(u) = \begin{cases} \cosh[(2n+1)u_0]\exp[-(2n+1)u] & \text{if } u>u_0, \\ \exp[-(2n+1)u_0]\cosh[(2n+1)u] & \text{if } u\leqslant u_0, \end{cases} \tag{38}$$

and

$$\alpha_n=\frac{4}{\pi}\frac{(-1)^n}{2n+1}\int_0^{\pi/2} \sin x \sin[(2n+1)x] \times\left[\frac{1+(M^2/\beta^2)\cos^2 x}{\cos^2 x+\beta^2\sin^2 x}\right]^{1/2} dx. \tag{39}$$

In the limit $M\to 0$ Eq. (37) gives

$$\psi(r,\theta) \overset{M\to 0}{\sim} \frac{S}{2}\begin{cases} \dfrac{\cos\theta}{r} & \text{if } r>1, \\ r\cos\theta & \text{if } r\leqslant 1. \end{cases} \tag{40}$$

This agrees with the $S\to 0$ limit of the low M solution established in Sec. III A, as it should. In the limit $M\to 1$ (but $M<1$)

$$\psi(r,\theta) \overset{M\to 1^-}{\sim} \frac{S}{\beta}\cos v \begin{cases} \exp(-u) & \text{if } u>u_0, \\ \cosh u & \text{if } u\leqslant u_0. \end{cases} \tag{41}$$

It is easy to show that this corresponds to a uniform flow of magnitude S/β inside the radius $r=1$. Since $\beta=\sqrt{1-M^2}$, there is a large amplification of the flow-through speed when M is close to 1.

For the parabolic potential, $\Omega=\Omega_{\text{par}}$, Eq. (31) describes the potential due to a charged cylinder of elliptic cross-section, $y^2+\beta^2\xi^2=1$, with a charge density per unit length of cylinder given by $\rho=(S\beta/2\pi)\xi$. We do not present the full solution in this case, but only the maximum x-velocity which is needed later. The velocity, $\nabla\psi$, is the ''electric field'' due to the hypothetical ''charge'' with the sign reversed. The maximum is clearly reached at the center and can be found by integrating the field contributions due to thin slices parallel to the y-axis. The result is

$$u_{\max}^{(\text{par})} \overset{S\to 0}{\sim} 1+\frac{4S}{\pi\beta}\int_0^{\pi/2} \cos\theta \sin\theta \tan^{-1}(\beta \tan\theta) d\theta. \tag{42}$$

When β is small, we can replace $\tan^{-1}(\beta \tan \theta)$ in the integrand by $\beta \tan\theta$. The approximation however, is not valid in a region of width of the order of β close to $\theta=\pi/2$. However, in this region, the actual integrand approaches zero whereas the integrand with the above approximation is of order β. The error in the integral is therefore of order β^2, and this is small, when β^2 is small, so that the approximation may be made for the whole interval $\pi/2>\theta>0$. The integral can now be easily evaluated and we obtain

$$u_{\max}^{(\text{par})} \overset{S\to 0}{\sim} 1+S \tag{43}$$

when $M \lesssim 1$.

We now show that in the linear hydrodynamics approximation, and in the absence of Landau damping, no beam deflection is possible as long as the incident flow is subsonic, in agreement with earlier observations.[8]

Theorem 2: In the limit of small S, the beam deflection is zero up to terms of order S provided $M<1$.

Proof: If we expand the density in an asymptotic series in the small parameter S and substitute in Eq. (2) we obtain $\rho=1+S\rho_1+\cdots$, where

$$\rho_1 = -\Omega - M^2 \frac{\partial \psi_0}{\partial x} = -\Omega - \frac{M^2}{\beta} \frac{\partial \psi_0}{\partial \xi}. \tag{44}$$

On substituting the expansion for ρ in the beam deflection formula (6), and, using the expression (32) for ψ, we obtain, to leading order in S;

$$a \frac{\rho_c}{\rho_*} \frac{d\alpha}{dz_*} = \frac{SM^2}{4\pi\beta^3 \int \Omega d\xi dy} \times \int \frac{\Omega_{\xi'}(\xi',y')\Omega_{\xi}(\xi,y)(\xi'-\xi)}{[(\xi-\xi')^2+(y-y')^2]} \times d\xi d\xi' dy dy'. \tag{45}$$

If we denote the four dimensional integral in the above expression by I, then, on interchanging the dummy variables ξ and ξ' we conclude that $I=-I$. Thus $I=0$, which implies the right hand side of the above expression is zero, and this proves the theorem. (Q.E.D.)

2. Supersonic flow

We consider Eq. (30) with $M>1$. Let us define $B=\sqrt{M^2-1}$. Then, in place of Eq. (31) we obtain the hyperbolic equation

$$\psi_{yy} - B^2 \psi_{xx} = S \frac{\partial \Omega}{\partial x}. \tag{46}$$

The characteristics of this equation are straight lines with slope $\pm 1/B$. Let us introduce oblique coordinates, (ξ,η), through the origin and parallel to the characteristics. The coordinate transformation laws follow from simple geometry;

$$x = \frac{B}{M}(\eta+\xi), \tag{47}$$

$$y = \frac{1}{M}(\eta-\xi), \tag{48}$$

and the inverse transformation

$$\xi = \frac{M}{2B}(x-By), \tag{49}$$

$$\eta = \frac{M}{2B}(x+By). \tag{50}$$

In the new coordinate system, Eq. (46) takes the form

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = -\frac{S}{2MB}\left(\frac{\partial \Omega}{\partial \xi} + \frac{\partial \Omega}{\partial \eta}\right), \tag{51}$$

which can be readily integrated;

$$\psi(\xi,\eta) = -\frac{S}{2MB}\left[\int_{-\infty}^{\eta} \Omega(\xi,\eta')d\eta' + \int_{-\infty}^{\xi} \Omega(\xi',\eta)d\xi'\right]. \tag{52}$$

When $M\to 1$, but $M>1$, the slope of the characteristics $\pm 1/B$, approach $\pm\infty$. The integrations along ξ and η in Eq. (52) get replaced by integrations along y, and we obtain

$$\psi(x,y) \overset{M\to 1^+}{\sim} -\frac{S}{2B}\int_{-\infty}^{+\infty} \Omega(x,y')dy'. \tag{53}$$

When $\Omega=\Omega_{\text{step}}$, the integral in Eq. (53) may be evaluated exactly, and we get

$$\psi_{\text{step}}(x,y) \overset{M\to 1^+}{\sim} -\frac{S}{B}\begin{cases}(1-x^2)^{1/2} & \text{if } |x|\leqslant 1,\\ 0 & \text{if } |x|>1.\end{cases} \tag{54}$$

The disturbance is confined to the band $|x|\leqslant 1$ and is solely in the x-direction. Clearly, the x-component of the velocity is singular along the lines $x=\pm 1$. If $\Omega=\Omega_{\text{par}}$, the integral (53) can also be evaluated in closed form and we get

$$\psi_{\text{par}}(x,y) \overset{M\to 1^+}{\sim} -\frac{2S}{3B}\begin{cases}(1-x^2)^{3/2} & \text{if } |x|\leqslant 1,\\ 0 & \text{if } |x|>1.\end{cases} \tag{55}$$

The disturbance is confined to $|x|\leqslant 1$ and is solely in the x-direction. However, unlike the case of the step potential there are no singularities in the velocity field.

When $M\to\infty$, the slope of the characteristics $\pm 1/B$, approach zero and $B\approx M$. The line integrals along the ξ and η axes reduce to integrals along the x-axis, so that Eq. (52) becomes

$$\psi(x,y) \overset{M\to\infty}{\sim} -\frac{S}{M^2}\int_{-\infty}^{x} \Omega(x',y)dx'. \tag{56}$$

When $\Omega=\Omega_{\text{step}}$, the integral in Eq. (56) may be evaluated exactly. We observe that the disturbance is confined within the region defined by $|y|<1$ and $x>-\sqrt{1-y^2}$, and, within this region we have:

$$\psi_{\text{step}}(x,y) \overset{M\to\infty}{\sim} -\frac{S}{M^2} \times \begin{cases} x+\sqrt{1-y^2} & \text{if } -\sqrt{1-y^2}<x<\sqrt{1-y^2},\\ 2\sqrt{1-y^2} & \text{if } x>\sqrt{1-y^2}.\end{cases} \tag{57}$$

In this case, the y velocity is singular along the lines $y\pm 1$. The integral in Eq. (56) may also be evaluated exactly when $\Omega=\Omega_{\text{par}}$. Again, we observe that the disturbance is confined within the region $|y|<1$ and $x>-\sqrt{1-y^2}$, and, within this region we have:

$$\psi_{\text{par}}(x,y) \overset{M\to\infty}{\sim} \frac{S}{M^2}\begin{cases}\frac{x^3}{3}-(1-y^2)x-\frac{2}{3}(1-y^2)^{3/2} & \text{if } -\sqrt{1-y^2}<x<\sqrt{1-y^2},\\ -\frac{4}{3}(1-y^2)^{3/2} & \text{if } x>\sqrt{1-y^2}.\end{cases} \tag{58}$$

As in the case of the $M\to 1$ limit, no singularities in the velocity field occur in the case of the parabolic potential.

We now compute the beam deflection due to the flow. We expand the density in the small parameter S, $\rho=1+S\rho_1+\cdots$, and substitute in Eq. (2) to derive

$$\rho_1=-\Omega-M^2\frac{\partial\psi_0}{\partial x}=-\Omega-\frac{M^3}{2B}\left(\frac{\partial\psi_0}{\partial\xi}+\frac{\partial\psi_0}{\partial\eta}\right). \tag{59}$$

On substituting this in the beam deflection formula (6), and using the transformation $dxdy=(2B/M^2)d\xi d\eta$ for coordinate differentials, we get the following expression for the beam deflection at leading order in S:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{SM^3}{16B^3\langle\Omega\rangle}\left\langle\frac{\partial\Omega}{\partial\xi}\int_{-\infty}^{\eta}\frac{\partial\Omega(\xi,\eta')}{\partial\xi}d\eta' + \frac{\partial\Omega}{\partial\eta}\int_{-\infty}^{\xi}\frac{\partial\Omega(\xi',\eta)}{\partial\eta}d\xi'\right\rangle, \tag{60}$$

where

$$\langle\cdot\rangle\equiv\frac{\int(\cdot)d\xi d\eta}{\int d\xi d\eta}=\frac{\int(\cdot)dxdy}{\int dxdy} \tag{61}$$

(the integrals are over the whole $x-y$ or $\xi-\eta$ plane). In deriving Eq. (60), we have dropped the terms $\Omega\Omega_\xi$, $\Omega\Omega_\eta$, $\Omega_\eta\int_{-\infty}^{\eta}\Omega_\xi d\eta'$ and $\Omega_\xi\int_{-\infty}^{\xi}\Omega_\eta d\xi'$ in the integrand as their contribution to the integral vanish on using integration by parts. Equation (60) can be written in an alternate form using the following identity

$$\int_{-\infty}^{+\infty}dxf(x)\int_{-\infty}^{x}f(x')dx'=\frac{1}{2}\left[\int_{-\infty}^{+\infty}f(x)dx\right]^2. \tag{62}$$

On transforming the numerator of Eq. (60) using the above identity, we get

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{SM^3}{32B^3}\frac{\int d\xi\left[\int d\eta\frac{\partial\Omega}{\partial\xi}\right]^2+\int d\eta\left[\int d\xi\frac{\partial\Omega}{\partial\eta}\right]^2}{\int d\xi d\eta\Omega} \tag{63}$$

(in the above expression, and throughout this paper, the interval of integration should be assumed $(-\infty,+\infty)$ when the integration limits are omitted), which shows clearly that the beam deflection is always positive.

In the limit $M\to\infty$, Eq. (63) has the asymptotic form

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to\infty}{\sim}\frac{\int dy\left[\int dx\frac{\partial\Omega}{\partial y}\right]^2}{4\int dxdy\Omega}\frac{S}{M^2}, \tag{64}$$

and in the limit $M\to 1$ (but $M>1$)

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to 1^+}{\sim}\frac{\int dx\left[\int dy\frac{\partial\Omega}{\partial x}\right]^2}{4\int dxdy\Omega}\frac{S}{\sqrt{M^2-1}}. \tag{65}$$

To prove this, we note, that the characteristic parallel to the η-axis for a given fixed value of ξ is at a distance $2B|\xi|/M^2$ from the origin. This follows from elementary geometry, knowing that the characteristics have slopes $\pm 1/B$. We define the new variable s equal to this distance in magnitude, and with the same sign as ξ;

$$s=\frac{2B}{M^2}\xi. \tag{66}$$

Clearly, (s,η) forms an orthogonal coordinate system. In terms of the variables s and η, the first term in the numerator of Eq. (63) may be written as

$$\int d\xi\left[\int d\eta\frac{\partial\Omega}{\partial\xi}\right]^2=\frac{2B}{M^2}\int ds\left[\frac{d}{ds}\int(\Omega)_s d\eta\right]^2, \tag{67}$$

where $(\Omega)_s$ indicates that s is being held constant in the integration. Now, we observe, that as $M\to\infty$, the η axis becomes congruent to the x-axis and the variable s therefore coincides with $-y$. Thus, Eq. (67) becomes

$$\int d\xi\left[\int d\eta\frac{\partial\Omega}{\partial\xi}\right]^2\overset{M\to\infty}{\sim}\frac{2}{M}\int dy\left[\frac{d}{dy}\int\Omega(x,y)dx\right]^2. \tag{68}$$

Similarly, in the limit $M\to 1^+$, $\eta\to y$ and $s\to x$ so that Eq. (67) becomes

$$\int d\xi\left[\int d\eta\frac{\partial\Omega}{\partial\xi}\right]^2\overset{M\to 1^+}{\sim}2B\int dx\left[\frac{d}{dx}\int\Omega(x,y)dy\right]^2. \tag{69}$$

A similar reduction can be done for the second integral in the numerator of Eq. (63) using the coordinates ξ and $s'=2B\eta/M^2$ as the basic variables. Using these asymptotic results in Eq. (63), and noting that $d\xi d\eta=(M^2/2B)dxdy$, Eqs. (65) and (64) easily follows.

In the special case of a radial potential $\Omega=\Omega(r)$, (63) can be greatly simplified. Clearly, for a radial potential, the integral $\int\Omega(\xi,\eta')d\eta'$ can only depend on the distance $|s|$, of the characteristic $\xi'=\xi$ from the origin. Hence, we may simplify the calculation by replacing this integral by a line integral along $y=|s|$:

$$\int_{-\infty}^{+\infty}\Omega(\xi,\eta')d\eta'=2\int_0^{\infty}\Omega(\sqrt{s^2+x^2})dx. \tag{70}$$

The ξ integration can now be performed by making the change of variables $\xi\to s$, and after some algebra we obtain the following remarkably simple formula for the beam deflection:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{\mathscr{K}S}{M\sqrt{M^2-1}}, \tag{71}$$

where

$$\mathscr{K}\equiv\frac{\int_0^{\infty}ds\left[\frac{d}{ds}\int_0^{\infty}\Omega(\sqrt{s^2+x^2})dx\right]^2}{\pi\int_0^{\infty}ds\Omega(s)s} \tag{72}$$

is a numerical coefficient determined solely by the structure of the potential Ω. For $M\to\infty$, Eq. (71) implies that the beam deflection $\sim\mathscr{K}S/M^2$, and for $M\to 1^+$, the deflection

$\sim \mathscr{K}S/\sqrt{M^2-1}$. These asymptotic results were already deduced in the last paragraph for a general potential, formula (71) is therefore consistent with previous results. The integral in Eq. (72) is readily evaluated for the parabolic potential, and we get

$$\mathscr{K}_{\mathrm{par}} = \frac{32}{15\pi} \approx 0.68. \tag{73}$$

For the Gaussian profile defined by

$$\Omega_{\mathrm{Gauss}}(r) = \exp(-r^2) \tag{74}$$

we get

$$\mathscr{K}_{\mathrm{Gauss}} = \left(\frac{\pi}{32}\right)^{1/2} \approx 0.31. \tag{75}$$

However, for the step potential, it is easy to show, that the integral in the numerator of Eq. (72) has a logarithmic divergence at $s=1$,

$$\mathscr{K}_{\mathrm{step}} = \infty. \tag{76}$$

This is not very surprising considering that the step potential implies an infinite force density on the ''rim,'' $r=1$, and we have already seen that the flow field around it also diverges on certain lines.

In hohlraum experiments, the flow across the laser beam is not constant, but will in general vary along the beam, that is, the Mach number M, is a function of z. One will then be interested in knowing the total deflection, $\delta\alpha$, suffered by the beam when it traverses a region of such flow. If we assume that the Mach number varies linearly over this region, the total deflection can be found by integrating Eq. (71) over the Mach number interval $(1,M)$;

$$\begin{aligned} a\frac{\rho_c}{\rho_*}\delta\alpha &= \left(\frac{dM}{dz_*}\right)^{-1} \int_1^M \frac{\mathscr{K}S}{M\sqrt{M^2-1}}\,dM \\ &= \left(\frac{dM}{dz_*}\right)^{-1} \mathscr{K}S \cos^{-1}\left(\frac{1}{M}\right). \end{aligned} \tag{77}$$

The maximum possible deflection is clearly

$$a\frac{\rho_c}{\rho_*}(\delta\alpha)_{\max} = \left(\frac{dM}{dz_*}\right)^{-1} \frac{\mathscr{K}S\pi}{2}. \tag{78}$$

Suppose $M=M(p)$ is the value of the Mach number for which the integrated deflection, Eq. (77), is a fraction p of the maximum deflection, Eq. (78). Then, it follows from Eqs. (77) and (78) that

$$M(p) = \sec\left(\frac{p\pi}{2}\right). \tag{79}$$

This equation, which is valid for any radial potential $\Omega=\Omega(r)$, may be useful for quickly identifying the Mach number range that produces the bulk of the deflection, so that, in a full-scale numerical simulation of realistic situations, the computational resources can be concentrated only in this parameter regime. A graph of $M(p)$ as a function of p is shown in Fig. 2. We would like to point out that even though the beam deflection formula has a singularity at $M=1$, the singularity is integrable, so that it is not necessarily true that most of the deflection comes from a small neighborhood of $M=1$. Indeed, it is evident from Eq. (79), that in order to obtain 50% of the total deflection one needs to consider Mach numbers up to 1.4 and to get 90% and 99% of the total deflection, the corresponding Mach numbers are 6.4 and 63.7 respectively [for the conditions assumed in Eq. (79), namely, linearly varying Mach number and constant background density, ρ_*].

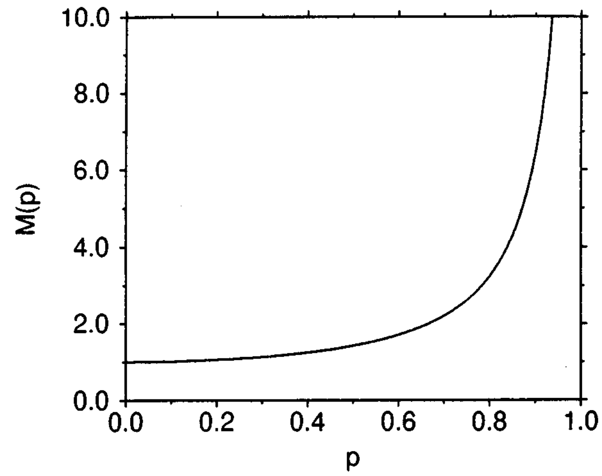

FIG. 2. The Mach number, $M(p)$, as a function of the fraction of total beam deflection, p, when the Mach number varies linearly with distance along the beam but the beam profile does not change.

The total deflection can be found in closed form for another situation of practical interest. This is the case where the flow Mach number is constant but the beam profile varies as in a single diffraction limited laser beam in the absence of self-focusing. Self-focusing may be neglected if the total power is small compared to a certain critical threshold. Further, even if the power is near or above the threshold, it has been observed in numerical simulations[3] that for large Mach numbers self-focusing is a small correction. Let $z_*=0$ denote the position of the focal plane. Then the variation of the beam intensity, which is proportional to $S(z_*)$, may be modeled as

$$\frac{S(z_*)}{S_0} = \left[1+\left(\frac{z_*}{2a_0F}\right)^2\right]^{-1}, \tag{80}$$

where S_0 and $a_0 \sim F\lambda$ are the values of the parameter S and the beam radius a at the focal plane. F is the F-number of the lens and λ is the wavelength of the laser. Equation (80) is an exact result for the intensity on the optic axis if the distribution of beam intensity across the lens is Gaussian. In the general case Eq. (80) is still qualitatively correct. The radius $a(z_*)$ then follows from the condition that the total power across the beam must be constant:

$$\frac{a(z_*)}{a_0} = \left[1+\left(\frac{z_*}{2a_0F}\right)^2\right]^{1/2}. \tag{81}$$

On substituting Eqs. (80) and (81) in Eq. (71), and integrating with respect to z_* from $-\infty$ to z_*, we obtain

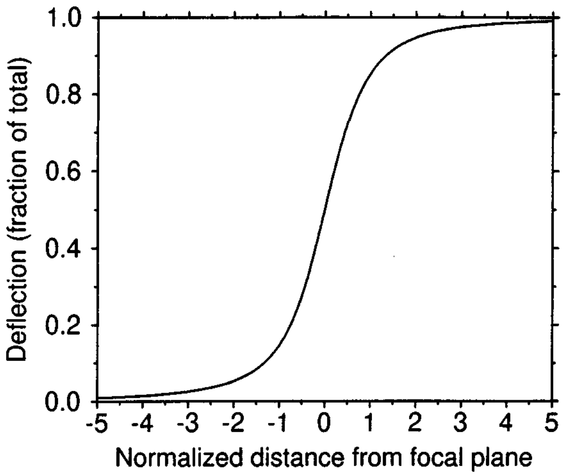

FIG. 3. The fraction of total deflection $\delta\alpha/(\delta\alpha)_{\max}$ as a function of dimensionless distance from the focal plane, $z_*/(2a_0F)$, for a single diffraction limited beam when the transverse flow does not vary along the beam.

$$\frac{\delta\alpha(z_*)}{(\delta\alpha)_{\max}} = \frac{1}{2}\left[1 + \frac{z_*/(2a_0F)}{\sqrt{1+\{z_*/(2a_0F)\}^2}}\right], \tag{82}$$

where the total deflection is given by

$$\frac{\rho_c}{\rho_*}(\delta\alpha)_{\max} = \frac{4S_0\mathscr{K}F}{M\sqrt{M^2-1}}. \tag{83}$$

A graph of $\delta\alpha/(\delta\alpha)_{\max}$ as a function of $z_*/(2a_0F)$ is shown in Fig. 3. It would be of interest to compare Eq. (83) with experimental work by Bauer in progress,[7] since preliminary estimates indicate that self-focusing is weak for several of the particular cases studied.

The existence of a nonzero drag, even in the absence of Landau damping, for supersonic flows, is the most significant result of this section. This is essentially the ''wave drag'' familiar in classical aerodynamic theory.[15] In linearized aerodynamic theory, a compressible inviscid fluid in potential flow does not exert any net force on a rigid obstacle as long as the inflow Mach number is less than one. However, when the inflow Mach number exceeds one, a finite drag appears. This ''wave drag'' approaches infinity near Mach number unity but decreases with increasing Mach number—a behavior that is responsible for the phenomenon of ''the sonic barrier'' in supersonic flight. The physical origin of this wave drag can be understood simply, from the energy conservation principle. When a fluid exerts drag on an obstacle, there is a reaction force that in turn does work on the fluid. If there is to be a steady state, this energy injected into the fluid has to be either dissipated in the fluid or transported to infinitely far points. Since Landau damping is neglected, and the equations have been linearized, which rules out shocks, there is no mechanism to dissipate energy. In subsonic flow the equations are elliptic in character so that any perturbations decay far from the source. Thus, energy cannot be removed to infinity. It follows, that the input of energy must also be zero, that is, the drag must be zero. When the Mach number exceeds one, the equation changes character to hyperbolic and the flow perturbations acquire finite values at infinity since they are propagated undiminished along the characteristics. This opens up a ''channel'' to transport energy to infinity and hence ''wave drag'' becomes possible. The existence of drag in this regime was explained in terms of an analogy with FBS (forward Brillouin scatter) by Hinkel *et al.*[8] The method of characteristics employed here illustrates that the basic mechanism is the ability to transmit momentum and energy ''to infinity,'' and that this ''wave drag'' is a universal mechanism that is well known in the classical literature on supersonic flows.

We conclude this section with the third in our series of ''null deflection'' theorems that complements the first two. It would appear, in the light of the above discussion, that the deflection should be zero in finite regions of parameter space for which the flow is subsonic everywhere. However, so far we have established this result only for $M \ll 1$ (Theorem 1) and for the subsonic part of the region $S \ll 1$ (Theorem 2). A stronger statement is indeed possible for potentials that are symmetric about the y-axis.

Theorem 3: If the potential, Ω, is symmetric about the y-axis, $\Omega(-x,y)=\Omega(x,y)$, and, if the following conditions are satisfied:

(a) All flow perturbations vanish infinitely far from the beam;
(b) There are no shocks anywhere in the flow field;
(c) A unique steady solution of the problem exists; then, the beam deflection must be exactly zero.

Proof: It follows from Eqs. (4) and (5) and the assumptions made in the theorem, that if $\phi(x,y)$ is a solution, then so is $-\phi(-x,y)$. It should be carefully noted that this conclusion would be false unless both the conditions (a) and (b) of the theorem were satisfied. Indeed, if there were shocks, then Eq. (4) is no longer valid over all space so that the above conclusion is not valid. Again, if the flow perturbations do not decay at infinity, $-\phi(-x,y)$ will not approach the constant flow of unit magnitude far upstream demanded by the boundary condition (5), and will not therefore be an acceptable solution. Now, it follows from (c) that the solution has the symmetry $-\phi(-x,y)=\phi(x,y)$, and this implies, by Eq. (2), that the density has reflection symmetry about the y-axis, $\rho(-x,y)=\rho(x,y)$. Since Ω and ρ are both symmetric functions of x, it follows that the integrand of the beam deflection formula (6) is an odd function of x so that the beam deflection must vanish. (Q.E.D.)

Note, that in Theorem 3, the conditions that there be no shocks and that the perturbations decay at infinity were postulated but it does not say in which regions of parameter space these conditions are fulfilled. The answer to this question is provided partly by Theorem 1 and Theorem 2 that guarantees these conditions when $M\to 0$ (S fixed) and $S\to 0$ ($M<1$ and fixed).

3. Transonic flow

The singularity in the beam deflection formula at $M=1$ indicates that the asymptotic expansion in S must become singular at $M=1$. This singularity arises because when the inflow Mach number is close to one, even a small perturbing potential may cause the flow to reach sonic speed somewhere in the flow domain:

$$M|\nabla\phi|=1. \tag{84}$$

The condition (84) defines a transition between regimes of elliptic and hyperbolic behavior for our basic equation (4).[16] If such a transition occurs in the flow domain, regions of elliptic and hyperbolic character would coexist in the flow field, with the possibility of a shock separating the two regions. This situation clearly cannot be described by the linearized Eq. (31) or (46) but terms beyond the first order in S need to be retained in the coefficient of ϕ_{xx}. This region of parameter space is called ''transonic.'' The boundary of the transonic regime in parameter space is defined by some critical curve $S=f_c(M)$ such that the linear theory gives qualitatively correct results for $S<f_c(M)$ but for $S>f_c(M)$ the solution exhibits mixed elliptic and hyperbolic character. Clearly, $f_c(M)\to 0$ as $M\to 1$, and in general, we expect $f_c(M)$ to increase as we move away from $M=1$ in either direction. It follows, from Theorem 3, that, for a symmetric beam profile, if $M<1$ and $S<f_c(M)$, there is no beam deflection, since the equations are elliptic in character so that perturbations decay far from the source, and, by definition, shocks do not appear until $S=f_c(M)$ is reached. Beam deflection may occur if either S goes above the critical curve $f_c(M)$ (shocks may appear) or if $M>1$ (equations become hyperbolic so perturbations propagate to infinity following characteristics). Thus, the boundary in parameter space separating the region in which beam deflection may occur from the region of null deflection is

$$S=\begin{cases} f_c(M) & \text{if } M\leqslant 1,\\ 0 & \text{if } M>1. \end{cases} \tag{85}$$

For a general potential, that is, one that is not necessarily symmetric about the y-axis, we have proved that Eq. (85) would still define the onset of beam deflection in the limits $M\ll 1$ (Theorem 1) and $M \lesssim 1$ (Theorem 2). When $1>M>0$, but not necessarily close to either of those limits, it seems plausible that the beam deflection would be zero as long as S is below the critical curve, Eq. (85), but we do not have a proof of this for an arbitrary Ω.

We now attempt to determine the critical curve $f_c(M)$. First, we examine the region $M\approx 1$. In this region, $S\ll 1$, so that $f_c(M)$ can be determined as the smallest S for a given value of M such that the condition (84) is reached at some point(s) in the flow field:

$$1-M^2-2M^2S(\partial_x\psi_0)=0. \tag{86}$$

This condition may be written in terms of $u_{\max}$ and $u_{\min}$ the maximum and minimum x-component of the total (that is, the sum of the inflow and the perturbation) velocity;

$$u_{\max}=1+\frac{\beta^2}{2M^2} \text{ if } M<1, \tag{87}$$

$$u_{\min}=1-\frac{B^2}{2M^2} \text{ if } M>1. \tag{88}$$

Consider the region $M\lesssim 1$. For the step potential, we know $u_{\max}$ from the solution (41). Thus, we can readily find $u_{\max}$, and hence the critical curve:

$$S=\frac{1}{2}(1-M^2)^{3/2}. \tag{89}$$

For the parabolic potential, in the same regime, $u_{\max}$ is given by Eq. (43), and hence we obtain:

$$S=\frac{1}{2}(1-M^2). \tag{90}$$

We now consider the supersonic branch $M\gtrsim 1$. It follows from the solution (54), that for the step potential, the x-velocity is $-\infty$ along the line $x=-1$. Thus, the criticality condition (88) is satisfied for arbitrarily small S. The critical curve for $M\gtrsim 1$ is therefore

$$S=0, \tag{91}$$

the linearized equations are never valid near $M=1$, no matter how small S might be. This result is of course to be expected, since the linear solution exhibits singularities. For the parabolic potential, we use the corresponding solution (55), to find $u_{\min}$. On substituting this into Eq. (88) we derive the following expression for the critical curve in the regime $M\gtrsim 1$:

$$S=\frac{1}{2}(M^2-1)^{3/2}. \tag{92}$$

We will now derive the asymptotic forms for the critical curve $S=f_c(M)$ in the regime where $M\ll 1$. Clearly, the condition for criticality in this case may be written as

$$Mu_{\max}=1. \tag{93}$$

On substituting $u_{\max}$, which is readily found from the solution (13), into Eq. (93), we derive the following equation for the critical curve in the $M\ll 1$ regime for the step potential:

$$2M=1+\exp(-S). \tag{94}$$

Equation (94) however shows that the critical curve asymptotes to $M=1/2$ as $S\to\infty$, so that our use of the $M\ll 1$ asymptotic results were not completely rigorous. However, (94) is still likely to be qualitatively correct, though deviations from it due to higher order corrections should be expected. The reason that the transonic regime does not occur for $M<1/2$ is that, for the step potential, the velocity remains finite as $S\to\infty$. As discussed in Sec. III A, when $S\to\infty$ the flow in the exterior of the beam is identical to potential flow around a rigid cylindrical obstacle, and in the interior, it is a uniform flow in the same direction as the inflow but of twice the magnitude. Thus, if $M<1/2$ sonic speed can never be reached even if S is arbitrarily large. A somewhat different result is obtained for the parabolic potential. The maximum velocity $u_{\max}$ is easily found from the solution (21), and we get the following equation for the critical curve:

$$S=\frac{1}{8M^2}. \tag{95}$$

In this case, $M\to 0$ as $S\to\infty$ so that the use of the $M\ll 1$ asymptotic results are fully justified. The essential difference between the parabolic and step potentials is that in the former case there is a small central region, of radius $\sim 1/\sqrt{S}$ where the velocity $\sim \sqrt{S}$ increases without bound. Thus, even for arbitrarily small M, there is an S for which the flow would eventually reach sonic speed at the center of the beam. This event may be marked by the onset of beam deflection due to the energy being dissipated in shocks that can appear near the center of the beam.

It is useful to have a single formula for the critical curve that reduces to the appropriate asymptotic forms at both the $M \to 0$ and $M \to 1^-$ limits. In the case of the step potential, the following composite formula may be constructed:

$$1-\exp(-S)=\frac{(1-M^2)^{3/2}}{2M}. \tag{96}$$

Note that Eq. (96) has not been "derived," and is not even unique. It is merely one possible "interpolation formula" between the two asymptotic results (94) and (89). When $S \to \infty$, Eq. (96) implies that $M \to M_c \approx 0.4$. This is slightly different from the limit $M_c = 0.5$ predicted by Eq. (94). However, since the asymptotic results for $M \to 0$ is not expected to apply strictly when $M \to M_c$, both the forms (94) and (96) are expected to be only qualitatively valid and the fact that they differ slightly near $M = M_c$ is not of much significance. A similar critical curve formula can be constructed for the parabolic potential in the subsonic regime:

$$\frac{S(4S+1)}{S+1}=\frac{1-M^2}{2M^2}. \tag{97}$$

Clearly, Eq. (97) reduces to Eq. (90) when $M \to 1^-$ and to Eq. (95) when $M \to 0$.

We summarize below the various asymptotic regimes:

Step potential:

$$1-\exp(-S)=\frac{(1-M^2)^{3/2}}{2M} \quad \text{if} \quad M<1,$$

$$S=0 \quad \text{if} \quad M>1. \tag{98}$$

Parabolic potential:

$$\frac{S(4S+1)}{S+1}=\frac{1-M^2}{2M^2}= \quad \text{if} \quad M<1,$$

$$S=\frac{1}{2}(M^2-1)^{3/2} \quad \text{if} \quad M>1. \tag{99}$$

Figure 4 shows the critical curves in parameter space $M-S$ obtained from Eqs. (98) and (99). It was seen in Sec. III B 2 that the supersonic beam deflection formula has a singularity at $M=1$. Figure 4 shows that the linear theory used in deriving this formula must break down close to $M=1$ for a fixed S, no matter how small. The nonlinearity characterizing this transonic regime is then expected to "regularize" the beam deflection formula at $M=1$. As of present, we have not succeeded in solving the necessary nonlinear equations that describe the transonic regime. There are very few analytical results in aerodynamics for transonic flows. The usual method[16] consists of making a transformation of variables so that the velocity (u,v) become the independent variables. This transformation, called the hodograph

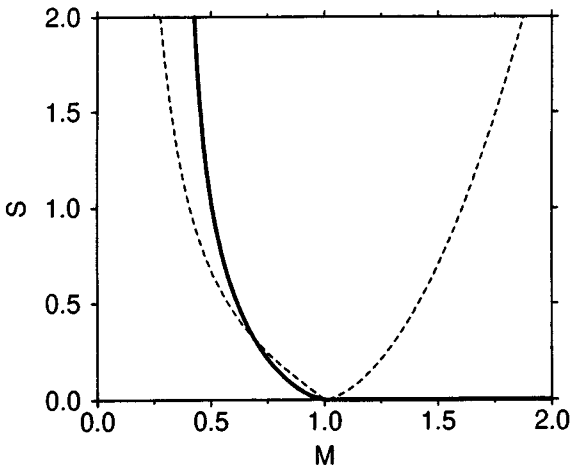

FIG. 4. The critical curve for the onset of the transonic regime, $S=f_c(M)$ (in the absence of Landau damping) for the step potential, Ω_{step} (solid line) and the parabolic potential, Ω_{par} (dashed line). When $M<1$, the beam deflection is zero if S is below the critical curve. For $M>1$, there is always beam deflection due to "wave-drag."

transformation, has the remarkable effect of making the equations linear. However, this only shifts the difficulty elsewhere—the boundary conditions now become equations of constraints that are very difficult to handle. In our problem, the hodograph transformation would make the potential Ω into a nonlinear function of the dependent instead of the independent variables. Nevertheless, it is conceivable that solutions can be found for some special classes of potentials, this is at the moment, an open problem.

A preliminary investigation of this transonic regime in the case of a one dimensional flow has been reported.[9] In this work the authors assume that the sonic point is located at the intensity maximum of the laser beam. It would appear that this corresponds to a very special curve on the parameter space $M-S$, viz., one that would rule out shocks. The generality of their results is therefore uncertain and it is not clear to what extent their conclusions can be applied to a real laser beam. It is interesting to note, that a closely related problem, in the context of air flow over mountains, has been solved in full generality.[17] However, though not without interest, solutions for one dimensional flows may be misleading because "wave drag" does not occur in one dimensional flows. Thus, the mechanism for beam deflection in supersonic flows discussed in Sec. III B 2 will be completely missed in the context of a model that allows flow in only one transverse dimension.

IV. BEAM DEFLECTION IN THE PRESENCE OF LANDAU DAMPING

The temperature and density in hohlraum plasmas are such, that molecular viscosity effects are quite negligible. Instead, the dominant dissipation mechanism is Landau damping. Since Landau damping is a collisionless dissipation mechanism, it does not appear explicitly in the moment equations of the Vlasov system. A proper treatment of Landau damping requires the solution of the Vlasov equation.

Attempting to understand the beam deflection effect through study of the Vlasov equation is however too ambitious for the present preliminary investigation. Instead, in this paper we will adopt a heuristic model, the rigorous justification of which will require a rederivation of the current results from the Vlasov equations.

We will use the model introduced by Rose,[3] which, in the steady state may be written, in our notation, as

$$M^2\partial_k(\rho u_i u_k) = -\partial_i\rho - S\rho\partial_i\Omega - 2M\gamma_0\int \frac{k_i k_j}{k}\hat{u}_j(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k}, \tag{100}$$

and

$$\partial_j(\rho u_j)=0. \tag{101}$$

Here a caret (ˆ) denotes Fourier-transform and summation is implied over repeated indices. The above model has the properties: (a) it is Galilean invariant (b) momentum is conserved in the absence of the ponderomotive force (c) it gives the correct dispersion relation for ion sound waves in the limit of linear theory. In Eq. (100), γ_0 is the dimensionless Landau damping parameter, so that the imaginary part of the acoustic frequency is given by $\gamma_0 c_s k$. In hohlraum applications γ_0 is typically in the range $0.01-0.1$. However, we will not make the assumption that γ_0 is small, most of the results in this section remain valid even when $\gamma_0\sim 1$. If γ_0 is assumed small in deriving a particular result, this fact will be made explicit at the appropriate point in the text. We now turn to the problem of determining what effect the Landau damping term has on the results related to beam deflection discussed in previous sections. The remainder of this section is presented in two subsections dealing with the $M\to 0$ and $S\to 0$ limits respectively.

A. Flow at low Mach number

In Sec. III A we proved that in the low Mach number limit, the beam deflection is zero. It was pointed out, that the physical reason behind this result was the absence of a mechanism to dissipate the energy gain that a beam deflection would imply. The presence of Landau damping however, does provide such a dissipation mechanism, so that in the presence of Landau damping we would expect the null deflection theorem (Theorem 1) of Sec. III A to be violated. In this section we show that this expectation is indeed true and derive a formula for the beam deflection.

We now introduce asymptotic expansions in the small parameter M, $\rho=\rho_0+M\rho_1+\cdots$, and $\mathbf{u}=\mathbf{u}_0+M\mathbf{u}_1+\cdots$. It is clear, from Eq. (100), that at the lowest order, the flow is Eulerian—the Landau damping term does not appear at this order. Thus, we may introduce the velocity potential ϕ_0, such that $\mathbf{u}_0=\nabla\phi_0$, and ϕ_0 satisfies Eq. (8). On substituting the asymptotic expansions in Eq. (100), we obtain Eq. (24) for ρ_0, and, for ρ_1 we obtain the equation

$$S^{-1}\nabla\rho_1+\rho_1\nabla\Omega-2\gamma_0\nabla\int \nabla\widehat{\Omega\cdot\nabla\phi_0}\,\frac{\exp(i\mathbf{k}\cdot\mathbf{x})}{k}d\mathbf{k}=0. \tag{102}$$

If we take the x-component of this equation, and integrate over all space, the first term drops out, since the density perturbation, which is proportional to the pressure perturbation, must vanish far from the origin in order to satisfy the boundary condition of uniform flow at infinity. Thus, we obtain,

$$\int \rho_1\Omega_x d\mathbf{x} = 2\gamma_0\int d\mathbf{x}\int d\mathbf{k}\frac{ik_x\exp(i\mathbf{k}\cdot\mathbf{x})}{k}\nabla\widehat{\Omega\cdot\nabla\phi_0}. \tag{103}$$

Clearly, the term ρ_0 has no contribution in the beam deflection formula (6), and, using Eq. (103), we get the following expression for beam deflection:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*} = \frac{M\gamma_0}{\int\Omega dxdy}\int d\mathbf{x}\int d\mathbf{k}\frac{ik_x\exp(i\mathbf{k}\cdot\mathbf{x})}{k}\nabla\widehat{\Omega\cdot\nabla\phi_0}. \tag{104}$$

We will derive an explicit expression for the beam deflection when the potential is radial, $\Omega=\Omega(r)$. In this case, we have the solution $\phi_0=A(r)\cos\theta$, where $A(r)$ satisfies Eq. (15). We substitute $\phi_0=A(r)\cos\theta$ in Eq. (104), and obtain after some simplification

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*} = \frac{M\gamma_0}{2\int_0^\infty\Omega(r)rdr}\int_0^\infty rdr\int_0^\infty dkW(k)kJ_0(kr), \tag{105}$$

where

$$W(k)\equiv\int_0^\infty dRRJ_1(kR)\Omega'(R)A'(R). \tag{106}$$

Here J_n is the Bessel function of order n, and, (r,θ) are polar coordinates of the point $\mathbf{x}$.

When $\Omega=\Omega_{\rm par}$, we are able to evaluate the integral analytically only in the limit of $S\gg 1$ or if $S\ll 1$. The latter simply reproduces the $M\ll 1$ limit of the results to be derived in Sec. IV B. Therefore we will only do the calculation when S is large. On substituting Eqs. (14) and (21) in Eq. (106) we get

$$W(k)=-\frac{2(1+2S)^{3/2}}{1+S}\int_0^1 dR\left[\frac{R^2}{(1+2SR^2)^{3/2}}\right]J_1(kR). \tag{107}$$

If we consider R as fixed, then in the limit of $S\to\infty$, the term in the square brackets reduces to $1/(2SR\sqrt{2S})$. Therefore, this approximation can be made in the integral for all R, except over a region of width $\sim 1/\sqrt{2S}$ near $R=0$. In this region, the approximate expression for the integrand will approach $J_1(kR)/(2SR\sqrt{2S})\approx k/(2S\sqrt{2S})$, while the integrand itself would approach zero. If the approximation is used throughout the interval, the error incurred in the integral would be $\sim k/(2S\sqrt{2S})\times 1/\sqrt{2S}=k/(4S^2)$. Therefore the relative error would be $\sim 1/\sqrt{S}$ and this is negligible when S is sufficiently large. Thus, on approximating the term in the square bracket by $1/(2SR\sqrt{2S})$, we get in the limit of large S

$$W(k)=-\frac{2}{S}\int_0^k\frac{J_1(x)}{x}dx. \tag{108}$$

On substituting the above expression for $W(k)$ in Eq. (105), we obtain, after some calculation involving integration by parts and standard properties[18] of Bessel functions:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{S\to\infty}{\sim}\frac{4M\gamma_0}{S}. \tag{109}$$

The most significant feature of the solution is that the beam deflection decreases with S for large S. When S is small, the deflection increases with S, as we show in the next section. An analogous calculation may be carried out for the beam deflection due to a step potential, Ω_{step}, and it is found that the beam deflection diverges, as we found in Sec. III B 2 for the $M>1$ and $S\ll 1$ regime.

The following physical argument for the inverse variation of the beam deflection with S is plausible. Beam deflection is a consequence of the scattering of fluid momentum by the ponderomotive potential, and is therefore expected to increase with S. However, the total momentum scattered cannot increase to infinity with S as even an impenetrable potential will scatter momentum at a rate at most $\sim aM^2$ (a is the linear dimension and M is the Mach number). The momentum scattered by an element of fluid which includes a localized potential is proportional to $d\alpha/dz_*$ times the intensity of the laser, that is, the scattered momentum $\sim S d\alpha/dz_*$. Since this quantity must saturate to a finite value as $S\to\infty$, the beam deflection $d\alpha/dz_*$ must decrease inversely with S. A note of caution should be added regarding the formula (109). In the limit of large S, the density fluctuation induced by the laser is no longer small compared to the background density. In this regime, our basic model Eq. (100) may not be accurate.[3] It would be of interest to check Eq. (109) against a calculation based directly on the Vlasov equation.

B. Flow past a weak ponderomotive potential

When S is small, we proceed by substituting the asymptotic expansions $\rho=1+S\rho_1+\cdots$, and $\mathbf{u}=\hat{\mathbf{x}}+S\mathbf{u}_1+S^2\mathbf{u}_2+\cdots$ into Eqs. (100) and (101). At lowest order in S, Eq. (100) gives an equation for determining $\mathbf{u}_1$. On taking the curl of this equation, we find $\nabla\times\mathbf{u}_1=0$, so that we may introduce the velocity potential ϕ_1 defined by $\mathbf{u}_1=\nabla\phi_1$. Equation (101) then gives

$$\frac{\partial\rho_1}{\partial x}+\nabla^2\phi_1=0. \tag{110}$$

Equation (100) gives at the first order in S

$$\rho_1+\Omega+M^2\frac{\partial\phi_1}{\partial x}+2M\gamma_0\int d\mathbf{k}|\mathbf{k}|\exp(i\mathbf{k}\cdot\mathbf{x})\widehat{\phi_1}(\mathbf{k})=0. \tag{111}$$

On transforming Eqs. (110) and (111) to Fourier-space and eliminating ρ_1, we obtain

$$\hat{\phi}_1=-\frac{ik_x\hat{\Omega}}{(k^2-M^2k_x^2+2iM\gamma_0k_xk)}. \tag{112}$$

To obtain the beam deflection, we substitute the asymptotic expansion for ρ in Eq. (6). The first nonzero contribution is due to ρ_1, which is obtained on using Eqs. (110) and (112)

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{S}{\int\Omega dxdy}\left[\int d\mathbf{k}\frac{2\pi^2ik_xk^2|\hat{\Omega}(\mathbf{k})|^2}{(k^2-k_x^2M^2+2iM\gamma_0k_xk)}\right]. \tag{113}$$

An alternate form is obtained by multiplying numerator and denominator by the complex conjugate of the denominator:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{SM\gamma_0}{\int\Omega dxdy}\left[\int_0^\infty dkk^2\int_0^\pi d\theta \times\frac{8\pi^2|\hat{\Omega}(k,\theta)|^2\cos^2\theta}{[(1-M^2\cos^2\theta)^2+4M^2\gamma_0^2\cos^2\theta]}\right]. \tag{114}$$

In the special case of radial potentials, $\Omega(r,\theta)=\Omega(r)$, the dependence of the beam deflection on M and γ_0 becomes independent of the shape of Ω. Indeed, for a radial potential, $\hat{\Omega}(\mathbf{k})=\hat{\Omega}(k)$, so that Eq. (114) reduces to

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=K_{\text{rad}}Sf(M,\gamma_0), \tag{115}$$

where

$$f(M,\gamma_0)\equiv\frac{2}{\pi}\int_0^{\pi/2}\frac{M\gamma_0\cos^2\theta}{(1-M^2\cos^2\theta)^2+4M^2\gamma_0^2\cos^2\theta}d\theta, \tag{116}$$

and

$$K_{\text{rad}}=\frac{4\pi^2\int_0^\infty dkk^2|\hat{\Omega}(k)|^2}{\int_0^\infty\Omega(r)rdr}. \tag{117}$$

It is easily shown that the function $f(M,\gamma_0)$ has the following asymptotic behavior as $\gamma_0\to 0$ in the case $M=1$ and $M<1$ respectively:

$$f(1,\gamma_0)\overset{\gamma_0\to 0}{\sim}\frac{1}{4\sqrt{\gamma_0}}, \tag{118}$$

and

$$f(M,\gamma_0)\overset{\gamma_0\to 0}{\sim}\frac{M\gamma_0}{2}+\frac{\gamma_0}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n!}\Gamma\left(n+\frac{3}{2}\right)M^{2n+1}, \tag{119}$$

where Γ denotes the Gamma-function.

We now examine the behavior of the beam deflection Eq. (114) for various limiting values of the parameters M and γ_0 in the case of an arbitrary (not necessarily radial) potential Ω.

Case A$(M\to 0)$: Since the integral in Eq. (114) is nonsingular when $M\to 0$, we may simply put $M=0$ in the integral. Thus we obtain

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to 0}{\sim}\frac{1}{2}K_0SM\gamma_0, \tag{120}$$

where

$$K_0\equiv\frac{16\pi^2\int_0^\infty dkk^2\int_0^\pi d\theta|\hat{\Omega}(k,\theta)|^2\cos^2\theta}{\int\Omega dxdy}. \tag{121}$$

Case B$(M\to 1)$: In this limit, we get from Eq. (114),

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to 1}{\sim}\frac{S\gamma_0}{\int\Omega\, dx dy}\left[8\pi^2\int_0^\infty dk k^2\int_0^\pi d\theta \times\frac{|\hat{\Omega}(k,\theta)|^2\cos^2\theta}{(\sin^4\theta+4\gamma_0^2\cos^2\theta)}\right], \tag{122}$$

which is in general a finite quantity. Thus, the singularity in the beam deflection formula is ''regularized'' by the introduction of Landau damping. Equation (122) can be simplified further if $\gamma_0 \ll 1$. In this case, we note that the contribution to the integral comes predominantly from the neighborhood of $\theta=0$ and $\theta=\pi$. Thus, $|\hat{\Omega}(k,\theta)|^2$ in the angular integral can be replaced by the constant value $|\hat{\Omega}(k,0)|^2=|\hat{\Omega}(k,\pi)|^2$. The integral is then evaluated by making a change of variables to $x=\cos\theta$ and linearizing the resulting integrand around $x=\pm 1$. This gives, for $\gamma_0 \ll 1$,

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to 1}{\sim}\frac{K_1 S}{4\sqrt{\gamma_0}}, \tag{123}$$

where

$$K_1=\frac{8\pi^3\int_0^\infty k^2|\hat{\Omega}(k,0)|^2 dk}{\int\Omega\, dx dy}. \tag{124}$$

Case C(M→∞): When $M\to\infty$, we make the change of variables $x=M\cos\theta$ in Eq. (114). The θ-integral then admits the following asymptotic approximation

$$\begin{aligned}\int_0^\pi d\theta\frac{|\hat{\Omega}(k,\theta)|^2\cos^2\theta}{[(1-M^2\cos^2\theta)^2+4M^2\gamma_0^2\cos^2\theta]} &= \frac{1}{M^3}\int_{-M}^{+M}\frac{x^2|\hat{\Omega}(k,\cos^{-1}(x/M))|^2}{(1-x^2)^2+4\gamma_0^2x^2} \\ &\quad\times\frac{dx}{(1-x^2/M^2)^{1/2}}\overset{M\to\infty}{\sim}\frac{1}{M^3}|\hat{\Omega}(k,\pi/2)|^2 \\ &\quad\times\int_{-\infty}^{+\infty}\frac{x^2}{(1-x^2)^2+4\gamma_0^2x^2}dx.\end{aligned} \tag{125}$$

In the passage to the limit on the last step, the singularities of the integrand at $x=\pm M$ were disregarded as these are integrable singularities—the contribution to the integral from the neighborhood of $x=\pm M$ approaches zero as $M\to\infty$ even though the integrand itself is singular. On substituting in Eq. (114) we obtain

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to\infty}{\sim}\frac{1}{2}SK_\infty\frac{g(\gamma_0)}{M^2}, \tag{126}$$

where

$$K_\infty\equiv\frac{8\pi^3\int_0^\infty k^2|\hat{\Omega}(k,\pi/2)|^2 dk}{\int\Omega\, dx dy}, \tag{127}$$

and

$$g(\gamma_0)\equiv\frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{\gamma_0 x^2}{(1-x^2)^2+4\gamma_0^2x^2}dx. \tag{128}$$

When γ_0 is small, the following approximation for $g(\gamma_0)$ is easily derived by linearizing the integrand around the points $x=\pm 1$ where most of the contribution to the integral arises,

$$g(\gamma_0)\approx 1. \tag{129}$$

If the potential is radial, $\Omega=\Omega(r)$, $K_0=K_1=K_\infty=K_{\rm rad}$. For the step or parabolic potentials an explicit evaluation is possible, and we get, after some calculation using standard properties of Bessel functions, $K_{\rm rad}(\text{step})=\infty$, $K_{\rm rad}(\text{parabolic})=64/(15\pi)\approx 1.36$.

Case D$(\gamma_0\to 0)$: A different analysis is required for $M<1$ and $M>1$. In the former case, the integrand in Eq. (114) is nonsingular when $\gamma_0\to 0$, so the asymptotic form is clearly

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{\gamma_0\to 0}{\sim}S\gamma_0 F_{\rm sub}(M), \tag{130}$$

where

$$F_{\rm sub}(M)=\frac{8\pi^2 M}{\int\Omega\, dx dy}\left[\int_0^\infty dk k^2\int_0^\pi d\theta\frac{|\hat{\Omega}(k,\theta)|^2\cos^2\theta}{(1-M^2\cos^2\theta)^2}\right]. \tag{131}$$

When $M>1$, the integrand develops a singularity at $\cos\theta=1/M$ when $\gamma_0\to 0$, so that a different analysis is required. In this case we use the alternate expression (113) for the beam deflection and rewrite it in the following form:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{4\pi^2 S}{\int\Omega\, dx dy}\Im\left[\int_{k_x\geqslant 0}dk_x\int dk_y \times\frac{k_x k^2|\hat{\Omega}(\mathbf{k})|^2}{(k^2-k_x^2M^2-2iM\gamma_0k_xk)}\right], \tag{132}$$

where $\Im$ denotes the imaginary part. We first perform the integral with respect to k_y holding k_x fixed. Since the denominator is $k_y^2-B^2k_x^2-2iM\gamma_0k_xk$ (we are using the notation $B=\sqrt{M^2-1}$ introduced earlier) and γ_0 is small, most of the contribution to the integral comes from the neighborhood of $k_y=\pm Bk_x$. Therefore, we put $k_y=\pm Bk_x+z$ and linearize with respect to z. Then the inner integral becomes

$$\frac{M^2}{2B}k_x^2\left[\int_{-\infty}^{+\infty}\frac{|\hat{\Omega}(k_x,Bk_x)|^2dz}{z-iM^2\gamma_0k_x/B}-\int_{-\infty}^{+\infty}\frac{|\hat{\Omega}(k_x,-Bk_x)|^2dz}{z+iM^2\gamma_0k_x/B}\right]. \tag{133}$$

We now use the standard result

$$\int_{-\infty}^{+\infty}\frac{dz}{z\mp i\epsilon}\overset{\epsilon\to 0}{\sim}\pm i\pi \tag{134}$$

to obtain the following limit when $\gamma_0\to 0$ and $M>1$:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{\gamma_0\to 0}{\sim}SF_{\rm sup}(M), \tag{135}$$

where

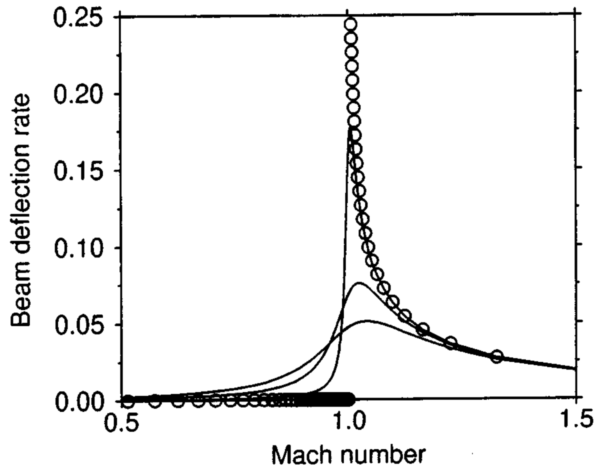

FIG. 5. The beam deflection for the Gaussian potential, Ω_{Gauss}, in the presence (solid lines) of Landau damping ($\gamma_0=0.01,0.05,0.1$) and in the absence (open circles) of Landau damping ($S=0.1$).

$$F_{\text{sup}}(M)\equiv\frac{M^2}{2B}\frac{4\pi^3}{\int\Omega dxdy}\left[\int_0^\infty k_x^2|\hat{\Omega}(k_x,Bk_x)|^2dk_x+\int_0^\infty k_x^2|\hat{\Omega}(k_x,-Bk_x)|^2dk_x\right]. \tag{136}$$

The most important difference between the subsonic [Eq. (130)] and supersonic [Eq. (135)] cases is that in the former, the beam deflection approaches zero as $\gamma_0\rightarrow 0$ but in the latter it approaches a finite value independent of γ_0. This is consistent with the results of Sec. III B for beam deflection in the absence of Landau damping. In particular, when $M\approx 1$, Eq. (136) reduces to

$$F_{\text{sup}}(M)\approx\frac{1}{\sqrt{M^2-1}}\frac{4\pi^3}{\int\Omega dxdy}\left[\int_0^\infty k_x^2|\hat{\Omega}(k_x,0)|^2dk_x\right]. \tag{137}$$

If we rewrite the Fourier-space integral as an integral over physical space, this is precisely the formula (65) for the supersonic beam deflection near $M=1$ in the absence of Landau damping. Similarly, when $M\gg 1$, Eq. (136) reduces (after making a change of variables from k_x to $k_y=Bk_x$) to

$$F_{\text{sup}}(M)\approx\frac{1}{M^2}\frac{4\pi^3}{\int\Omega dxdy}\left[\int_0^\infty k_y^2|\hat{\Omega}(0,k_y)|^2dk_y\right]. \tag{138}$$

If we rewrite the Fourier-space integral as an integral over physical space, this is precisely the formula (64) for the supersonic beam deflection for $M\gg 1$ in the absence of Landau damping.

Let us summarize the essential characteristics of the beam deflection that we learned from the various asymptotic limits presented above. First, the main effect of Landau damping is the regularization of the singularity in the beam deflection formula at $M=1$. When γ_0 is small, the beam deflection at $M=1$ is $\sim 1/\sqrt{\gamma_0}$ which is large, but finite. This is shown in Fig. 5, where we have plotted the beam deflection for the "Gaussian" potential, Eq. (74), in the presence and absence of Landau damping. In the former case we used Eq. (116), where $f(M,\gamma_0)$ was evaluated numerically and $K_{\text{rad}}=\sqrt{\pi/8}$ for the Gaussian potential. In the latter case we used the analytical formula (71). It must however be stressed that near $M=1$ we are in the "transonic" regime and nonlinear effects will also act to regularize the singularity by dissipating energy in shocks. The fact that Landau damping regularizes the singularity is qualitatively important but a proper quantitative theory must take into account the nonlinear effects together with Landau damping. Second, it is seen from Eqs. (126)–(129), when M is large, the beam deflection becomes asymptotically independent of γ_0, for small γ_0, and approaches the result we found earlier in the case of supersonic flow without Landau damping.

Beam deflection due to the "wave-drag" effect seems to have been observed in numerical simulations,[3,8] though the physical nature of the effect, in particular its persistence even in the limit of $\gamma_0=0$, and its relation to classical aerodynamic theory had not been realized. For transverse two dimensional flow past a laser beam, Hinkel *et al.* observe that the beam deflection occurs throughout the supersonic region. They also point out that this is in contrast to the corresponding case of a one-dimensional flow where the deflection occurs only near the sonic point. Both these results can be understood from our analysis. Since the beam deflection due to the "wave-drag" effect $\propto 1/M^2$ for large M, one would expect to observe beam deflection throughout the supersonic regime. Further, since "wave-drag" can occur in only two or three dimensional flows, in the case of a single spatial dimension, the only contribution to beam deflection is from the shocks in the transonic regime in the neighborhood of the sonic point, consistent with the observations of Hinkel *et al.* and the theoretical arguments of Kruer *et al.*[9] In the subsonic regime, for a fixed Landau damping (γ_0 fixed), Hinkel *et al.* suggest that the beam deflection in the regime of linear hydrodynamics $\sim M/(2-M^2)^2$, based on rough estimates from the beam deflection equation and numerical results. This is certainly true when M is small, but when $M\approx 1$, we have shown that for small Landau damping, the beam deflection has a "cusp" at $M=1$ which is of order $1/\sqrt{\gamma_0}$, and diverges as $\gamma_0\rightarrow 0$. The scaling proposed by the above authors would therefore seem inconsistent with this behavior, which is the expected "resonance" of linear theory. However, if we consider the alternate limit $\gamma_0\rightarrow 0$, but M is fixed and less than one, we have shown that the beam deflection $\propto f(M,\gamma_0)$ which is defined by Eq. (116). If we approximate the $\cos^2\theta$ in Eq. (116) by 1/2, we essentially recover the Hinkel *et al.* scaling $M\gamma_0/(2-M^2)^2$. This apparent contradiction is resolved if we observe that the limits $\gamma_0\rightarrow 0$, and $M\rightarrow 1$ do not commute. What this means, is that the scaling proposed by Hinkel *et al.* may be a good approximation if $M<1$ but as M approaches one, the error is of order $1/\sqrt{\gamma_0}$ which is large when $\gamma_0\ll 1$. That is, their approximation is not uniformly valid in the regime $0<M<1$. We would also like to point out that $f(M,\gamma_0)$ is given exactly by Eq. (119) when $M<1$, so one does not need to invoke, for potentials that are radial or random but statistically isotropic (see the next section), the estimate $f(M,\gamma_0)\sim M\gamma_0/(2-M^2)^2$ used by the above authors.

V. RANDOM PONDEROMOTIVE POTENTIALS

The intensity profile of laser beams in hohlraum experiments are not the smooth radially symmetric profiles studied in previous sections, but, rather, fluctuate randomly over the cross-section of the beam. The smooth radial ponderomotive potentials can be considered as reasonable models for only the envelope of the fluctuating intensity. There are two reasons for these fluctuations. First, in hohlraum experiments, before being focused by the lens, the beam is passed through a random phase plate (RPP). The RPP changes the phase of the wave at different locations on the cross-section of the incoming beam by random amounts. These individual ''beamlets'' interfere in the far field, creating an intensity pattern which varies rapidly in space. This scrambling of the phases is done to control certain laser plasma instabilities. The amplitude of the electric field at any point on the cross-section of such an RPP processed laser beam can be approximated fairly well by a random Gaussian process. The second mechanism for randomness in the beam intensity is the process of beam self-focusing. The statistics of such a self-focused beam is much more difficult to characterize. However, for a highly evolved self-focused beam, a reasonable model might be a collection of beamlets with identical beam profiles but different radii and intensities whose centers are scattered randomly in a uniform low intensity background. We have so far attributed a single length-scale, the macroscopic dimension a, to the laser beam. However, in the presence of random fluctuations, a second length-scale, the correlation length ℓ is introduced in the problem. It is clear from Eq. (6) that the beam deflection is controlled by the smallest scale of intensity variations in the beam. Thus, when $\ell \ll a$, which is usually the case, we expect the beam deflection will be determined by ℓ and will be independent of a. We will therefore assume in the following that the beam is of infinite cross-section and statistically homogeneous in the $x-y$ plane. However, if a is large enough, the flow gets excluded from the beam[3] over some length scale a_c due to nonlinear effects. Thus, ''infinite'' beam radius must be interpreted as $a_c \gg a \gg \ell$, so that ''flow exclusion'' effects can be ignored. As before, we divide the remainder of this section into two subsections dealing with the $M \ll 1$ and $S \ll 1$ cases respectively. The latter is further subdivided into two parts dealing with subsonic flow ($M<1$) and supersonic flow ($M>1$). We first consider the $S \ll 1$ case, since this is the regime for which we have more detailed results.

A. Flow past weak ponderomotive potentials

In this section, we present analytical results for beam deflection in the limit when the ponderomotive force is weak, $S \ll 1$. Explicit formulas are presented that give the beam deflection in terms of the Mach number and certain integrals of the correlation-function. Landau damping is neglected for the $M>1$ case, since, as has been shown in earlier sections, Landau damping in this case plays the role of a small correction serving mainly to regularize the singularity at $M=1$. Since the singularity at $M=1$ is integrable, the theory with no Landau damping gives reasonably accurate values for the integrated deflection. For the $M<1$ case we do take into account Landau damping, since, as was shown earlier, the beam deflection would be identically zero in this regime if Landau damping is neglected.

1. Supersonic flow

Since Eq. (60) in Sec. III B 2 was derived for a general potential $\Omega(x,y)$ it may be used when Ω is random. Equation (60) may be rearranged slightly and written in the form

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{SM^3}{16B^3}\frac{1}{\langle\Omega\rangle}\left[\int_{-\infty}^{\eta}d\eta' \times\left(\frac{\partial^2}{\partial\xi\partial\xi'}\langle\Omega(\xi,\eta)\Omega(\xi',\eta')\rangle\right)_{\xi'=\xi} +\int_{-\infty}^{\xi}d\xi'\left(\frac{\partial^2}{\partial\eta\partial\eta'}\langle\Omega(\xi,\eta)\Omega(\xi',\eta')\rangle\right)_{\eta'=\eta}\right]. \tag{139}$$

We now assume that the intensity fluctuations in the beam are statistically isotropic and introduce a correlation function $C(R)$ defined by

$$\langle\Omega(\xi,\eta)\Omega(\xi',\eta')\rangle=\langle\Omega^2\rangle C(R), \tag{140}$$

where the distance R between the points with the oblique coordinates (ξ,η) and (ξ',η') is given by

$$R^2=(\xi-\xi')^2+(\eta-\eta')^2+2\left(1-\frac{2}{M^2}\right)(\xi-\xi')\times(\eta-\eta'). \tag{141}$$

It is then easily shown using Eqs. (140) and (141) that

$$\left(\frac{\partial^2}{\partial\xi\partial\xi'}\langle\Omega(\xi,\eta)\Omega(\xi',\eta')\rangle\right)_{\xi'=\xi} =\langle\Omega^2\rangle\left[-C''(R)\left(1-\frac{2}{M^2}\right)^2+\frac{C'(R)}{R}\times\left\{\left(1-\frac{2}{M^2}\right)^2-1\right\}\right]. \tag{142}$$

On substituting this in Eq. (139) we obtain after several integration by parts the following expression for the beam deflection:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{S}{BM}\frac{\langle\Omega^2\rangle}{2\langle\Omega\rangle}\left[\int_0^{\infty}\frac{-C'(R)}{R}dR\right]. \tag{143}$$

In deriving Eq. (143) we have used the following properties of the correlation function: $C(0)=1$, which follows from its definition, $C'(0)=0$, which is a consequence of isotropy, and we take $C'(\infty)=0$. It is useful at this stage to introduce a correlation length ℓ defined by

$$\frac{1}{\ell}\equiv\int_0^{\infty}\frac{-C_*'(R_*)}{R_*}dR_*=\frac{1}{a}\int_0^{\infty}\frac{-C'(R)}{R}dR, \tag{144}$$

where, following our convention, we have used a suffix * to indicate that the variables are in physical dimensions. Thus, Eq. (143) may be written as

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{S}{M\sqrt{M^2-1}}\frac{\langle\Omega^2\rangle}{2\ell\langle\Omega\rangle}, \tag{145}$$

which shows explicitly that the only relevant length-scale is the correlation length ℓ. Note that Eq. (145) is identical to Eq. (71) with $\mathscr{K}=\langle\Omega^2\rangle/(2\ell\langle\Omega\rangle)$. The limits $M\to 1^+$ and $M\to\infty$ of Eq. (145) are of interest, and are easily derived:

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to 1^+}{\sim}\frac{S}{\sqrt{M^2-1}}\frac{\langle\Omega^2\rangle}{2\ell\langle\Omega\rangle}, \tag{146}$$

and

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}\overset{M\to\infty}{\sim}\frac{S}{M^2}\frac{\langle\Omega^2\rangle}{2\ell\langle\Omega\rangle}. \tag{147}$$

The correlation function $C_*(R_*)$ for a beam generated through RPP optics is derived in the Appendix, with the result

$$C_*(R_*)=\frac{1}{2}\left[1+\left(\frac{F\lambda}{\pi}\right)^2\frac{4}{R_*^2}J_1^2\left(\frac{\pi}{F\lambda}R_*\right)\right], \tag{148}$$

where λ is the wavelength of the laser, F is the F-number of the optic and J_1 is the Bessel function of order one. On substituting Eq. (148) in Eq. (144) and using some standard properties of Bessel functions we get

$$\ell=\frac{45}{64}F\lambda\approx 0.70F\lambda. \tag{149}$$

Thus, Eq. (145) gives for RPP optics the rather simple formula

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{64}{45}\frac{\langle\Omega\rangle}{F\lambda}\frac{S}{M\sqrt{M^2-1}}, \tag{150}$$

where we have used $\langle\Omega^2\rangle=2\langle\Omega\rangle^2$, as shown in the Appendix. This formula should be valid for all Mach numbers greater than unity except for a transonic region close to $M=1$ where it is expected to break down. Also, we would like to stress that the correlation function used in its derivation is valid as long as the statistics of the beam intensity has not been significantly altered by self-focusing effects. It has been observed in numerical simulations[10] that self-focusing tends to increase the relative proportion of high intensity regions in the beam. Thus, for a self-focused beam one would expect $\langle\Omega^2\rangle>2\langle\Omega\rangle^2$. Further, ℓ is decreased by self-focusing. Thus, Eq. (147) shows that self-focusing would tend to lead to an increase in the beam deflection.

2. Subsonic flow

In the subsonic case, the Landau damping must be included at the leading order, so we start from Eq. (113). On using the Fourier-space representation of Ω, we get

$$\frac{|\hat{\Omega}(\mathbf{k})|^2}{\int\Omega d\mathbf{x}}=\frac{1}{(2\pi)^4\int\Omega d\mathbf{x}}\int d\mathbf{x}d\mathbf{x}'\,\Omega(\mathbf{x})\Omega(\mathbf{x}')\times\exp[i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})]. \tag{151}$$

We now define $\mathbf{r}=\mathbf{x}'-\mathbf{x}$, replace the $\mathbf{x}$ integration by the average $\langle\rangle$ defined in Eq. (61), and introduce the correlation function $C(r)$ defined in Eq. (140) to derive

$$\frac{|\hat{\Omega}(\mathbf{k})|^2}{\int\Omega d\mathbf{x}}=\frac{1}{(2\pi)^4}\frac{\langle\Omega^2\rangle}{\langle\Omega\rangle}\int d\mathbf{r}C(r)\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{152}$$

Since $C(r)$ is independent of the direction of $\mathbf{r}$, the angular integration separates:

$$\frac{|\hat{\Omega}(\mathbf{k})|^2}{\int\Omega d\mathbf{x}}=\frac{1}{(2\pi)^3}\frac{\langle\Omega^2\rangle}{\langle\Omega\rangle}\int_0^\infty dr\,rC(r)J_0(kr). \tag{153}$$

We now substitute Eq. (153) in Eq. (114) and derive, after slight rearrangement:

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=S\frac{\langle\Omega^2\rangle}{\langle\Omega\rangle}f(M,\gamma_0)\times\int_0^\infty dk\,k^2\int_0^\infty dr\,rC(r)J_0(kr), \tag{154}$$

where $f(M,\gamma_0)$ is defined as in Eq. (116). We now use the property $(zJ_1(z))'=zJ_0(z)$ of Bessel functions[18] in Eq. (154) and integrate by parts. Next we replace $J_1(z)$ using $J_1(z)=-J_0'(z)$ and integrate by parts once more to derive

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=-S\frac{\langle\Omega^2\rangle}{\langle\Omega\rangle}f(M,\gamma_0)\int_0^\infty dk\int_0^\infty\frac{dz}{k}J_0(z)\times\left[C'\left(\frac{z}{k}\right)+\frac{z}{k}C''\left(\frac{z}{k}\right)\right]. \tag{155}$$

The k and z integrations can now be interchanged, and we obtain, since the integral of J_0 is one, the following rather simple result for the beam deflection:

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=S\frac{\langle\Omega^2\rangle}{\langle\Omega\rangle}\frac{1}{\ell}f(M,\gamma_0). \tag{156}$$

In Eq. (156), ℓ is the correlation-length defined by Eq. (144) and we have assumed, as before, that C and its derivatives decay sufficiently rapidly at infinity to justify all the mathematical operations used in the derivation of Eq. (156).

It should be noted that $M<1$ was not assumed in the derivation of Eq. (156). Thus, Eq. (156) is equally valid for $M>1$, and one can show, following the analysis of Sec. IV B, that the two expressions (156) and (145) are equivalent when $\gamma_0\to 0$ and M is fixed and greater than one. When $M<1$, $f(M,\gamma_0)$ is given by the series (119) if γ_0 is small. For RPP optics, ℓ is given by Eq. (149), and, $\langle\Omega^2\rangle=2\langle\Omega\rangle^2$, so that, in this case we derive the continuation of formula (150) into the subsonic regime:

$$\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{128}{45}\frac{\langle\Omega\rangle}{F\lambda}Sf(M,\gamma_0). \tag{157}$$

B. Flow at low Mach number

In the case $M\ll 1$, we first show how the beam deflection problem can be translated into a problem in the theory of random media. We note that Eq. (104) may be written as

$$a\frac{\rho_c}{\rho_*}\frac{d\alpha}{dz_*}=\frac{M\gamma_0}{\langle\ln\kappa\rangle}\left\langle\frac{\partial}{\partial x}\int d\mathbf{k}\frac{\exp(i\mathbf{k}\cdot\mathbf{x})}{k}\widehat{\left(\frac{\nabla\kappa\cdot\nabla\phi_0}{\kappa}\right)}\right\rangle. \tag{158}$$

In writing Eq. (158), we have invoked the ''dielectric analogy'' introduced in Sec. III A with $\kappa\equiv\exp(-S\Omega)$ as the ''dielectric coefficient'' of that equivalent problem. Thus, the problem translates to the following: if a dielectric medium with randomly varying dielectric coefficient $\kappa(x,y)$, with given statistical properties, is placed in a uniform electric field of unit magnitude, calculate the quantity (158) where ϕ_0 is the electric potential that satisfies Eqs. (11) and (9). This is a problem in the theory of random media. We do not have a solution to this problem at this time. However, we are able to solve the simpler problem of determining the average flow inside the beam for some idealized models of beam profile by making use of known results in random media theory. This is done in the following two paragraphs.

We first consider the case of a strongly self-focused beam. Let us suppose that all the energy in the beam is concentrated in hot-spots of radius a_0 distributed randomly inside the beam of radius a with number density N_c. The fractional area covered by these hot spots, $N_c\pi a_0^2$ is assumed to be small, but $a\gg a_0$, so that the total number of hot spots is very large. We will assume that all the hot spots are of identical structure and intensity, though this restriction can be easily removed by replacing N_c by $N_c^{(i)}$, the concentration of spots of type i, in the final result and summing over i. We first show, that as a first approximation, the mean flow inside the beam can be considered uniform with Mach number M which is related in a simple way to the Mach number M_0 of the external flow incident on the beam. To show this, we consider the equivalent problem in the theory of random dielectric media, with M reinterpreted as the mean electric field and M_0 as the external field. The relation between M and M_0 is then a standard derivation of that subject, where it is shown[19] that

$$M=\frac{M_0}{\kappa_m}, \tag{159}$$

where κ_m is a constant. The quantity κ_m is called the ''effective dielectric constant.'' For a dilute distribution of cylindrical inclusions of dielectric constant κ_2 in a medium with dielectric constant κ_1, it can be shown[19] that

$$\kappa_m=\kappa_1+2N_c\pi a_0^2\kappa_1\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}, \tag{160}$$

where N_c is the number of inclusions per unit area. If instead of a constant κ, the inclusions have a distribution $\kappa(x,y)$ localized in a region of radius a_0, formula (160) can still be considered approximately valid if κ_2 is regarded as some averaged dielectric constant. In fact, the three dimensional version of Eq. (160) is often used as a model for computing the macroscopic dielectric constant of a medium, regarding the molecules as spheres of a uniform dielectric. In that subject, the three dimensional equivalent of Eq. (160) is known as the ''Clausius-Mossotti relation'' or as the ''Lorentz-Lorenz formula'' [Eq. (160) is strictly equivalent to the Clausius-Mossotti relation if in addition to replacing the cylindrical inclusions by spheres, one assumes $\kappa_2-\kappa_1\ll\kappa_1$]. In the present model of a strongly self-focused laser beam, $\kappa_1=1$ and $\kappa_2=\exp(-gS)$, where g is some number of order unity determined by the shape of the potential Ω. In particular, for the step potential, $g=1$. Thus, we get the following formula for the Mach number inside the beam

$$M=\frac{M_0}{[1-2N_c\pi a_0^2\tanh(gS/2)]}, \tag{161}$$

where N_c is the number of hot spots per unit area, a_0 is the hot spot radius, and M_0 is the Mach number of the external flow. The notable feature of Eq. (161) is that M actually increases as S increases, and, as $S\to\infty$ (with a_0 and N_c fixed) it saturates at the value $M=M_0/(1-2N_c\pi a_0^2)$. This is a consequence of the fact mentioned earlier, that unlike a physical dielectric, our hypothetical dielectric medium has a negative susceptibility. For a beam undergoing self-focusing, the total power is conserved, so that the appropriate limit would be $S\to\infty$, $N_ca_0^2\sim 1/S$. In this limit, $M\to M_0$ as the self-focusing increases.

When the beam is not self-focused, as for example immediately upon entering the plasma, the above model clearly cannot be applied. In this situation, we sketch briefly an alternate model for which κ_m can be computed. We suppose that the potential Ω can assume only one of two values, Ω_1 or Ω_2 ($\Omega_2>\Omega_1$). These values can be distributed quite arbitrarily in the $x-y$ plane with only the following restrictions:

(a) The surface separating these two phases should have a well defined normal at every point (except on a set of measure zero).
(b) The resulting distribution should be statistically isotropic.
(c) If the two phases Ω_1 and Ω_2 are interchanged, all statistical properties of the distribution remain invariant.

The assumptions (a) and (b) are perfectly reasonable for the class of physical problems of interest. The assumption (c) is a reasonable model provided self-focusing is not important, for in the presence of self-focusing, there is a clear asymmetry between regions of low and high intensity. The two-phase model itself is of course an abstraction to make the problem tractable. In a real laser beam Ω will have a continuous spectrum of values. The numbers Ω_1 and Ω_2 can be related to the mean, $\langle\Omega\rangle$, and variance $\langle(\Delta\Omega)^2\rangle$ in the real beam if we require that the hypothetical two phase medium have the same mean and variance as the real one, and, that the two phases occur with equal probability. We then get the following two equations for determining Ω_1 and Ω_2:

$$\Omega_1+\Omega_2=2\langle\Omega\rangle, \tag{162}$$

$$(\Omega_1-\langle\Omega\rangle)^2+(\Omega_2-\langle\Omega\rangle)^2=2\langle(\Delta\Omega)^2\rangle, \tag{163}$$

with the solution ($\Omega_2>\Omega_1$ by convention)

$$\Omega_1=\langle\Omega\rangle-\langle(\Delta\Omega)^2\rangle^{1/2}, \tag{164}$$

$$\Omega_2=\langle\Omega\rangle+\langle(\Delta\Omega)^2\rangle^{1/2}. \tag{165}$$

In order to have $\Omega_1 \geqslant 0$, we need to assume $\langle (\Delta\Omega)^2 \rangle^{1/2} \leqslant \langle \Omega \rangle$. For a beam focused through RPP optics and in the absence of significant self-focusing, $\langle \Omega^2 \rangle = 2\langle \Omega \rangle^2$ (see Appendix), that is, $\langle (\Delta\Omega)^2 \rangle^{1/2} = \langle \Omega \rangle$. Thus, $\Omega_1 = 0$ and $\Omega_2 = 2\langle \Omega \rangle$ for RPP optics.

To calculate κ_m, we now utilize a generalization due to Mendelson,[20] of a result due to Keller.[21] It was shown by the above authors, that for a two phase medium characterized by dielectric constants (or electrical or thermal conductivities) κ_1 and κ_2 satisfying assumption (a), the following relation is true

$$\kappa_m^{xx}(\kappa_1,\kappa_2)\kappa_m^{yy}(\kappa_2,\kappa_1) = \kappa_1\kappa_2. \tag{166}$$

In Eq. (166), κ_m^{xx} and κ_m^{yy} are the principal values of the effective dielectric tensor, and the interchange of κ_1 and κ_2 in the argument list indicates that the two phases in the medium have been interchanged. Now, if the two phase medium also satisfies conditions (b) and (c) of our model, we get from Eq. (166), as has been pointed out by Mendelson,

$$\kappa_m = \sqrt{\kappa_1\kappa_2}. \tag{167}$$

On substituting the appropriate values for our model, $\kappa_1 = \exp(-S\Omega_1)$ and $\kappa_2 = \exp(-S\Omega_2)$ in Eq. (167), we obtain

$$\kappa_m = \exp(-S\langle \Omega \rangle), \tag{168}$$

where use has been made of Eq. (162). To obtain the mean Mach number inside the beam we substitute Eq. (168) in Eq. (159) to obtain

$$M = M_0 \exp(S\langle \Omega \rangle). \tag{169}$$

Thus, once again we observe that the flow inside the beam is amplified, but this time the amplification factor increases exponentially with the intensity of the beam with no possibility of saturation. It seems reasonable to conclude, as a general qualitative statement, that the flow tends to be amplified inside the beam when self-focusing is weak, but as the beam becomes strongly self-focused, the flow speed once again decreases and asymptotes to the value outside the beam. The above conclusions are of course based on linear theory. In reality, nonlinear effects will cause ''flow exclusion''[3] and this will tend to decrease the Mach number inside the beam. It would be interesting to see whether these qualitative effects can be observed in numerical simulations.

VI. SUMMARY AND CONCLUSIONS

Let us summarize the main conclusions of this paper. We considered the two-dimensional flow of a plasma transverse to a laser beam characterized by a certain distribution of ponderomotive force which was assumed known. The beam deflection was obtained by computing the momentum lost by the fluid. Except for this calculation of beam deflection, the back reaction of the flow in changing the characteristics of the laser beam was completely neglected. The problem, thus limited, can be characterized by three dimensionless parameters. They are, a nondimensional parameter characterizing the strength of the beam, S, the flow Mach number, M, and the dimensionless Landau damping coefficient γ_0.

We first considered the two-parameter problem by neglecting the Landau damping, $\gamma_0 = 0$. If $M < 1$, we showed that there cannot be any beam deflection unless S exceeds a certain critical value, $S = f_c(M)$. This critical curve $f_c(M)$ decreases monotonically with M, approaches zero as $M \to 1$, and asymptotically approaches infinity for some value $M = M_c$, where M_c can be zero or have a finite positive value depending on the nature of the beam profile. The physical reason for the existence of such a critical curve is that beam deflection implies a finite drag on the flow and this in turn requires a mechanism to either dissipate energy, or transport it to infinity. When $M < 1$, the governing equations are elliptic, perturbations decay far from the source so that energy cannot be transported to infinity. Further, in the absence of Landau damping, there is no physical dissipation mechanism. Therefore the drag is zero. The critical curve corresponds to the onset of possible shock formation in the flow, thereby allowing a dissipation mechanism and hence beam deflection. When $M > 1$, the equations are hyperbolic, so that energy can be propagated to infinity along characteristics, and we show that in this regime there is always beam deflection. Explicit formulas for the beam deflection are presented when $S \ll 1$, which show that, the rate of bending of the beam is proportional to $1/\sqrt{M^2 - 1}$ when M is close to, but greater than one, and inversely proportional to M^2 when M is large. The critical curve $f_c(M)$ continues for $M > 1$ but in this regime it corresponds to the breakdown of linear theory rather than the onset of beam deflection. If $S > f_c(M)$, the equations are mixed in character containing regions of elliptic flow embedded in the overall hyperbolic flow with the possibility of shocks separating the two regions. We call this the ''transonic'' regime and the linearized theory for $S \ll 1$ can no longer be applied. The singularity at $M = 1$ falls within this transonic regime and would be ''regularized'' by nonlinear effects. The most important feature of this singularity is that it is integrable. As a result, the total deflection across a region where M increases linearly is not localized in a small neighborhood of $M = 1$, as has been presumed by some researchers. In fact, we show, that 50% of the total deflection occurs at $M = 1.4$ and 90% occurs at $M = 6.4$. When the singularity is regularized by nonlinear effects, these values of M will only increase.

We then examined how the above results are modified in the presence of Landau damping, $\gamma_0 > 0$. Landau damping provides a dissipation mechanism, so that for $M < 1$, beam deflection is possible even below the critical curve, $S < f_c(M)$. Explicit solutions were presented for some model potentials when $M \ll 1$. A result that provides some physical insight is that the beam deflection increases linearly when S is small, but when S is large, it decreases in inverse proportion to S. The other significant effect of Landau damping is that the singularity at $M = 1$ is regularized. This result is significant but not quite rigorous because a proper treatment of the $M = 1$ case must take into account nonlinear effects in the transonic regime that were wholly ignored in this paper. However, for small enough S, the effect of Landau damping is expected to dominate the corrections due to nonlinear hydrodynamics as observed in numerical simulations due to Rose.[10] The boundary in γ_0–S space that corresponds to the transition between the two regimes is not expected to be a sharp one and has not yet been determined. Further, if the beam is of sufficiently large spatial extent, flow exclusion,[10] which is a strictly nonlinear effect may become important even if S is arbitrarily small. For $M>1$, the presence of Landau damping does not result in any qualitative change in the Mach number dependence of the beam deflection. Indeed, in the limit $M\rightarrow\infty$, the beam deflection with and without Landau damping are asymptotically equal if γ_0 is small.

We then considered the more realistic situation where the beam intensity profile has random fluctuations. We presented explicit formulas for the beam deflection when $S\ll 1$, for both, the subsonic ($M<1$) as well as the supersonic ($M>1$) cases. The randomness of the beam intensity enters into these formulas only through a certain integral involving the two point correlation function of the intensity. For a beam focused through RPP optics and in the absence of significant self-focusing, we derived an explicit expression for the two point correlation function. We used this to derive formulas (150) and (157). In the case of $M\rightarrow 0$, but S fixed we could only derive a few concrete results. First, it was shown that in this limit, the problem of beam deflection is formally equivalent to a problem in the theory of a random dielectric medium. However, we were not able to solve this latter problem. It follows from this analogy that the mean flow inside the beam is related in a simple way to the incident flow through the ''effective dielectric constant.'' We used known results for this constant and succeeded in deriving expressions for the mean flow for two simplified limiting models. The first model corresponds to a strongly self-focused beam and the second relevant to a situation where self-focusing is negligible. In either case we showed that the collective effect of the nonuniformities inside the beam result in a larger mean flow inside the beam than outside of it.

We consider this paper to be a very preliminary investigation of a complex problem, and much remains to be done. Our assumption that the fluid flow interacts with a known beam profile is of course only the first step towards investigation of the full problem, in which the flow and the beam modify each other. This is the theory of nonlinear self-focusing in the presence of a flow. Some numerical studies on this are available but it would be of interest to investigate if this can be supplemented by a theoretical study along the lines of the present paper. We pointed out that our asymptotic theory for weak laser beams break down in the neighborhood of Mach number one. This is the transonic region and the equations in this regime are intrinsically nonlinear in character. Even though the singularity in the beam deflection formula at Mach number one, is integrable, our theory should be considered incomplete without a proper treatment of the transonic regime. The effect of Landau damping is included in this paper through a model that is ad hoc except in the limit of linear hydrodynamics. It is well known that Landau damping is a collisionless dissipation mechanism and does not appear as an explicit term in the moment equations of the Vlasov system. It would therefore be satisfying to be able to rederive some of the results in this paper related to Landau damping starting directly from the Vlasov equation. This remains an open problem. In the case of a random potential at low Mach numbers, we showed, that the problem becomes identical to that of a ''random media'' problem. However, we did not succeed in exploiting this analogy to any great extent. This is another area of possible future work. The essentially nonlinear phenomenon of flow ''braking'' or ''exclusion'' due to removal of flow momentum is an area where more careful analysis will be needed. Finally, in this paper we hardly touched the issue of stability of the various steady flow solutions that were presented. In preliminary numerical investigations that we have conducted, we have seen some evidence of instabilities. The analysis of stability of flows around laser beams remains an open problem, and little is known about this subject.

This work was primarily motivated by a problem of technological interest, namely, the proper aiming of laser beams at targets in hohlraum experiments. It is therefore reasonable to ask to what extent is this work useful for that technology. Some of the formulas we derived, such as the ones presented in Sec. V A may be directly useful in applications where self-focusing might be negligible. These closed form expressions may also be useful as benchmarks for testing numerical codes designed for more complex situations. The parameter space is so large in problems involving laser plasma interactions, that it is difficult to make much progress in either numerical or experimental work without a very good physical insight. In this work we attempted to contribute in some way towards development of such insight through simple analytically solvable models. Even though one would most likely need to turn to numerical simulation and experiments in most practical applications, the physical insight gained from analysis of the kind presented here may be valuable in guiding such work and interpreting their results.

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APPENDIX: THE TWO POINT CORRELATION FOR RPP OPTICS

We will derive an expression for the two point correlation function of the ponderomotive potential in the situation where the laser beam is brought to a focus after being filtered by a RPP. Consider the focus of the lens as the origin and the z-axis as directed away from the lens, so that the $x-y$ plane is the focal plane. We take the beam to be plane polarized in the x-direction to simplify the algebra but it is trivial to prove that the expression derived for the correlation function is true independent of the polarization. The general time dependent electric field may then be written as

$$\mathbf{E}(\mathbf{x},t)=\hat{\mathbf{x}}\Re[\epsilon\exp\{i(kz-\omega t)\}], \tag{A1}$$

where $\Re$ indicates ''real part,'' and the complex amplitude, ϵ, varies rapidly in x and y but only weakly in z. Since $\Omega \propto \overline{E^2}=|\epsilon|^2/2$ (where the overbar denotes time average), we have

$$C(r) \equiv \frac{\langle \Omega(\mathbf{x}) \Omega(\mathbf{y}) \rangle}{\langle \Omega^2(\mathbf{x}) \rangle} = \frac{\langle \epsilon(\mathbf{x}) \epsilon^*(\mathbf{x}) \epsilon(\mathbf{y}) \epsilon^*(\mathbf{y}) \rangle}{\langle |\epsilon|^4 \rangle}, \tag{A2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors in the $x-y$ plane and $r \equiv |\mathbf{x}-\mathbf{y}|$. For a random phase plate, the complex amplitude may be written as

$$\epsilon(\mathbf{x}) = \epsilon_0 \sum_{\mathbf{k}} \exp[i(\mathbf{k} \cdot \mathbf{x} + \phi_{\mathbf{k}})], \tag{A3}$$

where ϵ_0 is a constant, $\phi_{\mathbf{k}}$ are the random phases introduced by the RPP, $\mathbf{k}$ is the $x-y$ component of the wave vector in the contribution from a single element of the RPP, $\mathbf{x}$ is a point on the focal plane and the summation is over all elements of the RPP. For the present derivation we only require that these phases satisfy the following two relations

$$\langle \exp[i\phi_{\mathbf{k}}] \rangle = 0, \tag{A4}$$

and

$$\langle \exp[i\{\phi_{\mathbf{k}} - \phi_{\mathbf{k}'}]\rangle = \delta_{\mathbf{k},\mathbf{k}'}. \tag{A5}$$

In particular, Eqs. (A4) and (A5) are satisfied if $\phi_{\mathbf{k}}$ assumes the values 0 and π with equal probability, and, the phases of different elements are uncorrelated. On substituting Eq. (A3) in Eq. (A2), and using the properties (A4) and (A5), we easily derive

$$C(r) = \frac{1}{2}\left[1 + \left|\frac{1}{N}\sum_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{r})\right|^2\right], \tag{A6}$$

where N is the total number of RPP elements. The summation in Eq. (A6) goes over all wavevectors with magnitude less than $k_{\max} = k_0 \sin\alpha$, where α is the angle subtended by the lens diameter at the focus and $k_0 = 2\pi/\lambda$, λ being the wavelength of the laser. Introducing the F-number by $F = f/D$, the ratio of the focal length, f, to the diameter, D, we have $k_{\max} \approx \pi/\lambda F$. When N is large, the sum in Eq. (A6) may be replaced by an integral, and we get on using standard properties of Bessel functions[18]

$$C(r) = \frac{1}{2}\left[1 + \frac{4}{r^2 k_{\max}^2} J_1^2(r k_{\max})\right]. \tag{A7}$$

When r is large, $\Omega(\mathbf{x})$ and $\Omega(\mathbf{y})$ are uncorrelated, so that combining Eqs. (A2) and (A7), we get $\langle \Omega^2 \rangle = 2\langle \Omega \rangle^2$.

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